Serial Dictatorship Mechanisms with Reservation Prices*

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February 16, 2019

Abstract

We propose a new set of mechanisms, which we call *serial dictatorship mechanisms with* individual *reservation prices* for the allocation of homogeneous indivisible objects, e.g., specialist clinic appointments. We show that a mechanism φ satisfies *minimal tradability*, *individual rationality*, *strategy-proofness*, *consistency*, *independence of unallocated objects*, and *non wasteful tie-breaking* if and only if there exists a reservation price vector r and a priority ordering $\succ$ such that φ is a *serial dictatorship mechanism with reservation prices* based on r and $\succ$. We obtain a second characterization by replacing *individual rationality* with *non-imposition*. In both our characterizations r, $\succ$, and φ are all found simultaneously and endogenously from the properties. Finally, we illustrate how our model, mechanism, and results, capture the normative requirements governing the functioning of some real life markets and the mechanisms that these markets use.

JEL classification: C78, D47, D71

Keywords: serial dictatorship; individual reservation prices; strategy-proofness; consistency.

*We thank the editor (Nicholas Yannelis), an anonymous referee, David Delacrétaz, Fuhito Kojima, and Steven Williams, for helpful discussions, comments and suggestions.

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1 Introduction

In many markets the resources to be allocated are overdemanded. The rationing does not always happen through market mechanisms, which leads to inefficiencies, i.e., the objects or services are typically not allocated to the agents who value them, and can pay, the most. In such markets, priority orderings (e.g., consumer queues, waiting lists, and so on) often emerge as the primary criteria for rationing the demand, with payments being only secondary. Given the private and social costs of the inherent inefficiencies, *why do such markets exist*?

One possible explanation is that priority orderings may be preferred because they capture social values such as *egalitarianism* and *orderliness* (Mann, 1969). When objects or services to be allocated are some form of basic needs, priority orderings may be regarded as a *just* procedure (Konow, 2003). Along the same lines, recent experiments show that agents' preferences extend beyond their own allocation and payments; the mechanism that generates the outcomes is important too, and agents also value the *procedural fairness* that comes with priority orderings (Dold and Khadjavi, 2017).

In this paper, we provide *normative justifications* for the existence of such markets and we show that a set of formal normative criteria can be used to derive both, priorities over agents and individual reservation prices, while simultaneously pinning down a new mechanism (which, as we discuss in Section 5, mimics a mechanism in use in real medical markets) that combines agents' priorities and individual reservation prices.

Our model is as follows. There are a set of homogeneous indivisible objects and a set of potential agents. Agents' preferences over receiving an object and their own payment are represented by general utility functions that are not necessarily quasilinear, and agents cannot trade or make transfers among themselves. A *mechanism* allocates the objects to the agents and specifies payments for all agents, i.e., it selects an *outcome*. We consider mechanisms that satisfy desirable normative criteria. Intuitively, these criteria are as follows. *Minimal tradability* requires that the object is allocated to some agent at least for some utility profile. *Individual rationality* ensures that all agents voluntarily participate. *Non-imposition* is a weakening of *individual rationality* specifying that agents who do not value the object cannot be forced to make a positive payment. *Strategy-proofness* guarantees that no agent can profitably misreport his valuation for the object. *Consistency* requires that given an outcome, if some agents leave with their allotments, then the outcome for all remaining agents remains the same as before. With *independence of unallocated objects*, if not all objects are allocated, then removing unallocated objects leaves the outcome unchanged. *Non wasteful* *tie-breaking* requires that agents are not indifferent between [receiving the object and paying for it] and [not receiving the object and not paying anything].

Given a *priority ordering* $\succ$ over the set of potential agents that arranges them in a queue and a *reservation price vector* r that specifies an individual reservation price for each agent, the associated *serial dictatorship mechanism with reservation prices* works as follows. First, an object is offered to the agent with the highest priority. If he chooses to take it, he pays his reservation price. If no more objects are left, all other agents receive and pay nothing, and we stop. Otherwise, an object is offered to the agent with the second highest priority. If he chooses to take it, he pays his reservation price. If no more objects are left, all other agents receive and pay nothing, and we stop. Otherwise, we continue until either all objects are assigned, or all (finitely many) agents have been offered an object. Note that if the reservation prices for all agents are zero, our mechanism essentially reduces to the classical *serial dictatorship mechanism*.

Our main result is that a mechanism φ satisfies *minimal tradability*, *individual rationality*, *strategy-proofness*, *consistency*, *independence of unallocated objects*, and *non wasteful tie-breaking* if and only if there exists a reservation price vector r and a priority ordering $\succ$ such that φ is a *serial dictatorship mechanism with reservation prices* based on r and $\succ$ (Theorem 1). We also obtain two other related characterizations: first, for single object allocation problems, *independence of unallocated objects* can be dropped from Theorem 1 (Corollary 1) and second, we can replace *individual rationality* with *non-imposition* in Theorem 1 (Corollary 2). Note that in our characterizations, neither the individual reservation prices nor the priority ordering the mechanism is based on are assumed as primitives; instead, the prices and priorities are derived, i.e., found endogenously, from the normative criteria, together with the serial dictatorship mechanism with reservation prices based on them. We show that all the normative criteria that we use in our characterizations are logically independent, confirming that each criterium is indispensable (Appendix C). Finally, we give examples of real-life settings, such as the allocation of the next-available consultant-led medical appointments in public hospitals in Australia, for which our assumptions and modeling are well-suited, and for which our results provide valuable insights (Sections 5 and 6).

The remainder of this paper is organized as follows. Next, we review the related literature. In Section 2, we introduce the model and the axioms, that is, the normative criteria that we use. In Section 3, we introduce serial dictatorship mechanisms with reservation prices. In Section 4, we present our characterization results. In Section 5, we present a real life example that is closely related to our work. In Section 6, we conclude.

Related Literature

Our model and our serial dictatorship mechanisms with individual reservation prices are new. For *house allocation problems* (Hylland and Zeckhauser, 1979) in which, unlike in our model, objects are heterogeneous and there are no reservation prices or payments, several characterizations of classical serial dictatorship mechanisms are available. Svensson (1999) shows that a mechanism is *strategy-proof*, *non-bossy*, and *neutral*, if and only if it is a serial dictatorship. Ergin (2000) shows that a mechanism is *weakly Pareto optimal*, *pairwise consistent*, and *pairwise neutral*, if and only if it is a serial dictatorship. Ehlers and Klaus (2007) show that if a mechanism satisfies *Pareto optimality*, *strategy-proofness*, and *consistency*, then there exists a priority structure such that the mechanism "adapts to it." For a model in which indivisible objects need to be allocated among agents who have responsive preferences and who each have a quota that must be filled exactly, Hatfield (2009) shows that the only *Pareto optimal*, *strategy-proof*, *non-bossy*, and *neutral* mechanisms are serial dictatorships. Note that with the exception of *strategy-proofness*[1] and *consistency*, the properties used in all the characterizations above are different from ours; moreover, even *strategy-proofness* and *consistency* are substantially different from ours due to the obvious differences in the modeling. Various other notable modifications of the house allocation model and serial dictatorship mechanisms have been proposed in the literature, but these are further away from our model and from the classical serial dictatorship mechanism we relate to.[2] We discuss similarities and differences between our *serial dictatorship mechanism with reservation prices* and its properties and the *second price auction* and its properties in Section 6.

Finally, our work can also be thought of as being related to the characterizations of deferred acceptance mechanisms (Kojima and Manea, 2010; Ehlers and Klaus, 2014, 2016) or immediate acceptance mechanisms (Kojima and Ünver, 2014; Doğan and Klaus, 2018); the commonality being that in those characterizations the priorities (or more generally, choice functions) are obtained from the mechanism using a set of normative criteria in the same spirit in which in our characterizations the reservation prices and priorities are derived from the mechanism using a set of properties.

[1] *Strategy-proofness* is a key property that is "obviously" satisfied – in the sense of Li (2017) – by all the classical serial dictatorship mechanisms and by our own serial dictatorship with reservation prices.

[2] For instance, restricted endowment inheritance mechanisms introduced by Pápai (2000) and characterized by Pápai (2000) and Ehlers et al. (2002) are essentially serial dictatorships where in each iteration, we might have either single or twin dictators. For a model where one good needs to be assigned among agents who all have single-dipped preferences, Klaus (2001a,b) provides characterizations of mechanisms that for the largest part have serial dictatorship components.

2 Model

We consider the situation where a non-negative number of homogeneous indivisible objects can be allocated to a set of agents; the number of objects and the set of agents can change. Let $\mathbb{N}$ be the set of potential agents and $\mathcal{N}$ be the set of all non-empty finite subsets of $\mathbb{N}$, $\mathcal{N} \equiv \{N \subseteq \mathbb{N} : 0 < |N| < \infty\}$.[3]

For any set of agents $N \in \mathcal{N}$ and any non-negative number of objects k, an *allocation vector* $a = (a_i)_{i \in N} \in \{0,1\}^N$ such that $\sum_{i \in N} a_i \leq k$ describes which agents in N receive an object; we allow for the possibility that only some objects or no object is allocated. We denote the *set of allocation vectors* for a set of agents $N \in \mathcal{N}$ and a number of objects $k \in \mathbb{Z}_+$ by

$$\mathcal{A}(N,k) \equiv \left\{ a = (a_i)_{i \in N} : a \in \{0,1\}^N \text{ and } \sum_{i \in N} a_i \leq k \right\}.$$

We assume that an agent $i \in \mathbb{N}$ may have to pay a non-negative *price* $p_i \in \mathbb{R}_+$, and we denote the *set of payment vectors* for a set of agents $N \in \mathcal{N}$ by

$$\mathcal{P}(N) \equiv \left\{ p = (p_i)_{i \in N} : p \in \mathbb{R}_+^N \right\}.^4$$

We assume that agents only care about receiving an object or not and their own payment. Each agent $i \in \mathbb{N}$ has preferences that are: (i) strictly decreasing in the price paid; (ii) such that given the same price, receiving an object is weakly better than not receiving it; and (iii) either there exists a price which makes the agent indifferent between [receiving an object at this price] and [not receiving it and paying nothing], or he strictly prefers to [obtain an object, whatever the price] over [not receiving it and paying nothing]. Formally, we represent an agent i's preferences ($i \in \mathbb{N}$) by a utility function $u_i : \{0,1\} \times \mathbb{R}_+ \to \mathbb{R}$ that satisfies the following three properties:

(i) if $0 \leq p'_i < p_i$, then $u_i(0,p'_i) > u_i(0,p_i)$ and $u_i(1,p'_i) > u_i(1,p_i)$;

(ii) for each $p_i \geq 0$, $u_i(1,p_i) \geq u_i(0,p_i)$; and

(iii) either there exists a price v_i such that $u_i(1,v_i) = u_i(0,0)$, or for each $p_i \geq 0$, we have $u_i(1,p_i) > u_i(0,0)$ and $v_i \equiv \infty$; v_i is agent i's *valuation* of the indivisible object.[5]

[3] A finite set of potential agents would not change any of our results.

[4] Setting the set of payment vectors equal to the Cartesian product of a discrete or finite price set would not change any of our results. As we discuss in detail in Section 5, ruling out negative payments or transfers is natural in certain contexts of rationing, e.g., when allocating appointments for certain medical services.

[5] Requiring continuity of u_i would be a less general assumption that guarantees the existence of valuation v_i.

An example of an agent i's preferences with valuation v_i are quasilinear preferences u_i defined for each $(a_i, p_i) \in \{0,1\} \times \mathbb{R}_+$ by $u_i(a_i, p_i) = v_i a_i - p_i$.

We denote the *set of utility profiles* for a set of agents $N \in \mathcal{N}$ by

$$\mathcal{U}(N) \equiv \{u = (u_i)_{i \in N} : \text{ for each } i \in N,\ u_i : \{0,1\} \times \mathbb{R}_+ \to \mathbb{R} \text{ satisfies (i), (ii), and (iii)}\}$$

and the associated *set of valuation vectors* by

$$\mathcal{V}(N) \equiv \left\{v = (v_i)_{i \in N} : \text{ for each } i \in N,\ v_i \in \mathbb{R}^N \cup \{\infty\}\right\}.$$

A *problem* (of allocating k objects among a group of agents) γ is a triple $(N, u, k) \in \mathcal{N} \times \mathcal{U}(N) \times \mathbb{Z}_+$. We denote the *set of all problems* for $N \in \mathcal{N}$ and $k \in \mathbb{Z}_+$ by $\Gamma(N, k)$.

An *outcome* for any problem $\gamma \in \Gamma(N, k)$ consists of an allocation vector $a \in \mathcal{A}(N, k)$ and a payment vector $p \in \mathcal{P}(N)$. We denote the *set of outcomes* for a problem $\gamma \in \Gamma(N, k)$ by

$$\mathcal{O}(N, k) \equiv \mathcal{A}(N, k) \times \mathcal{P}(N).$$

A *mechanism* φ is a function that assigns an outcome to each problem. Formally, for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, and each $\gamma \in \Gamma(N, k)$, $\varphi(\gamma) \in \mathcal{O}(N, k)$. Note that we can also represent a mechanism φ by its *allocation rule* α and *payment rule* π, i.e., for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, and each $\gamma \in \Gamma(N, k)$, $\alpha : \Gamma(N, k) \to \mathcal{A}(N, k)$, $\pi : \Gamma(N, k) \to \mathcal{P}(N)$, and $\varphi(\gamma) = (\alpha(\gamma), \pi(\gamma))$. We denote the *allotment* of agent i at outcome $\varphi(\gamma)$ by $\varphi_i(\gamma) = (\alpha_i(\gamma), \pi_i(\gamma))$.

Given $N \in \mathcal{N}$, a vector $x \in \mathbb{R}^N$, and $M \subseteq N$, let x_M denote the vector $(x_i)_{i \in M} \in \mathbb{R}^M$. It is the restriction of vector x to the subset of agents M. We also use the notation $x_{-i} = x_{N \setminus \{i\}}$. For example, $(\bar{x}_i, x_{-i})$ denotes the vector obtained from x by replacing x_i with $\bar{x}_i$. We use corresponding notational conventions for utility profiles.

Properties of Mechanisms

Our first property ensures that (i) if there are at least as many agents as objects, then there is some utility profile for which all objects are allocated and (ii) if there are more objects than agents, then there is some utility profile at which all agents receive an object.

Minimal Tradability: A mechanism φ satisfies *minimal tradability* if for each $N \in \mathcal{N}$ and each $k \in \mathbb{Z}_+$,

(i) for $k \leq |N|$, there exists a utility profile $u \in \mathcal{U}(N)$ such that $\sum_{i \in N} \alpha_i(N, u, k) = k$ and

(ii) for $k > |N|$, there exists a utility profile $u \in \mathcal{U}(N)$ such that $\sum_{i \in N} \alpha_i(N, u, k) = |N|$.

Our *minimal tradability* coincides with Sakai's (2013) for single object problems.

The following property allows agents who have no value for objects to withdraw from the problem at no cost (i.e., these agents cannot be forced to pay a positive price for an object).

Non-Imposition: A mechanism φ satisfies *non-imposition* if for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, each $\gamma \in \Gamma(N, k)$, and each $i \in N$, if u_i is such that $v_i = 0$, then $\pi_i(\gamma) = 0$.

Non-imposition was first introduced by Sakai (2008) for single object problems; he also observed that this property is very weak as it is satisfied by virtually all of the auction mechanisms in the literature.

For $N \in \mathcal{N}$ and $k \in \mathbb{Z}_+$, an outcome $(a, p) \in \mathcal{O}(N, k)$ is *individually rational* for utility profile $u \in \mathcal{U}(N)$ if for each $i \in N$, $u_i(a_i, p_i) \geq u_i(0, 0)$. Equivalently, an outcome $(a, p) \in \mathcal{O}(N, k)$ is individually rational for utility profile $u \in \mathcal{U}(N)$ with associated valuation vector $v \in \mathcal{V}(N)$ if for each $i \in N$, **(IR1)** $[a_i = 0$ implies $p_i = 0]$ and **(IR2)** $[a_i = 1$ implies $p_i \leq v_i]$. By requiring the mechanism to only choose individually rational outcomes we express the idea of voluntary participation.

Individual Rationality: A mechanism φ satisfies *individual rationality* if for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, and each $\gamma \in \Gamma(N, k)$, $\varphi(\gamma)$ is an *individually rational* outcome.

Strategy-proofness requires that no agent can benefit from misrepresenting his preferences.

Strategy-Proofness: A mechanism φ satisfies *strategy-proofness* if for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, each $(N, u, k) \in \Gamma(N, k)$, each $i \in N$, and each u'_i such that $u' \equiv (u'_i, u_{-i}) \in \mathcal{U}(N)$, $u_i(\varphi_i(N, u, k)) \geq u_i(\varphi_i(N, u', k))$.

That is, a mechanism is *strategy-proof* if (in the associated direct revelation game) it is a weakly dominant strategy for each agent to report his utility function truthfully.

Lemma 1. *The following relations among properties hold:*

(a) *individual rationality implies non-imposition;*

(b) *strategy-proofness and non-imposition imply individual rationality.*

Lemma 1 is a generalization of results for quasilinear utility functions and single object problems due to Sakai (2013, Proposition 1 (ii) and (iii)). We prove the lemma in Appendix A.

Consistency, first introduced by Thomson (1983), is one of the key properties in many frameworks with variable populations.[6] Adapted to our setting, *consistency* requires that if some agents leave with their allotments, then the allocation and the payments for all remaining agents should not change. Let $N \in \mathcal{N}$, $k \in \mathbb{Z}_+$, $\gamma = (N, u, k) \in \Gamma(N, k)$, and $M \subseteq N$. Now, when the set of agents M leaves problem γ with their $\sum_{i \in M} \alpha_i(\gamma)$ allotted objects, there are $k_{N \setminus M} = k - \sum_{i \in M} \alpha_i(\gamma)$ objects left. Hence, the *reduced problem* is $\gamma_{N \setminus M} = (N \setminus M, u_{N \setminus M}, k_{N \setminus M})$.

Consistency: A mechanism φ satisfies *consistency* if for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, each $\gamma \in \Gamma(N, k)$, and each $M \subseteq N$, we have $\varphi(\gamma_{N \setminus M}) = \varphi(\gamma)_{N \setminus M}$.

Next, we require that if not all objects are allocated, removing unallocated objects leaves the outcome unchanged.

Independence of Unallocated Objects: A mechanism φ satisfies *independence of unallocated objects* if for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, and each $\gamma \in \Gamma(N, k)$, we have $\varphi(N, u, k) = \varphi(N, u, \sum_{i \in N} \alpha_i(\gamma))$.

Our next property excludes that the mechanism selects outcomes where the agent who receives the object is indifferent between his allotment and not receiving the object at price zero. The idea behind this property is to not wastefully assign the object to such an agent because another agent might prefer to receive it. In that sense, *non-wasteful tie-breaking* is a mild efficiency requirement.

Non Wasteful Tie-Breaking: A mechanism φ satisfies *non wasteful tie-breaking* if for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, each $\gamma \in \Gamma(N, k)$, and each $i \in N$, $\alpha_i(\gamma) = 1$ implies that $u_i(1, \pi_i(\gamma)) \neq u_i(0, 0)$.

3 Serial Dictatorships with Reservation Prices

In order to define a serial dictatorship with individual reservation prices, we first need to fix *reservation prices* and a *priority ordering.*

We assume that for each agent $i \in \mathbb{N}$ a *(fixed) reservation price* $f_i \geq 0$ exists. We interpret f_i as the smallest price at which the object can be allocated to agent i. We denote a *vector*

[6] Thomson (2015) provides an extensive survey of *consistency* in various applications.

of (fixed) reservation prices for the set of potential agents $\mathbb{N}$ by $f = (f_i)_{i \in \mathbb{N}}$ and by $\mathcal{F}$ we denote the *set of all (fixed) reservation price vectors* for $\mathbb{N}$.

A *priority ordering* $\triangleright$ over the set of potential agents $\mathbb{N}$ is a complete, asymmetric, and transitive binary relation, with the interpretation that for any two distinct agents $i, j \in \mathbb{N}$, $i \triangleright j$ means that i has a higher priority than j. Let $\mathcal{P}$ denote the *set of all priority orderings* over $\mathbb{N}$.

Given a reservation price vector $f \in \mathcal{F}$ and a priority ordering $\triangleright \in \mathcal{P}$, the *serial dictatorship mechanism with reservation prices* based on f and $\triangleright$ is denoted by $\psi^{(f,\triangleright)}$ and determines an outcome for each problem $\gamma = (N, u, k) \in \Gamma(N, k)$ with associated valuation vector $v \in \mathcal{V}(N)$ as follows.

Step 0: If $k = 0$, then stop and all agents receive and pay nothing. Otherwise, continue.

Step 1: The agent with the highest priority in N is considered. Let $i \in N$ be this agent.
- If $v_i > f_i$, then agent i obtains the object and pays f_i. Set $k_1 := k - 1$. If $k_1 = 0$, then we stop and all remaining agents receive and pay nothing. Otherwise, continue.
- If $v_i \leq f_i$, then agent i receives and pays nothing. Set $k_1 := k$ and continue.

Step l: The agent with the l^{th} highest priority in N is considered. Let $j \in N$ be this agent.
- If $v_j > f_j$, then agent j obtains the object and pays f_j. Set $k_l := k_{l-1} - 1$. If $k_l = 0$, then we stop and all remaining agents receive and pay nothing. Otherwise, continue.
- If $v_j \leq f_j$, then agent j receives and pays nothing. Set $k_l := k_{l-1}$ and continue.

We continue until either all objects are allocated or all agents have been considered.

Formally, for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, and each $\gamma = (N, u, k) \in \Gamma(N, k)$ with associated valuation vector $v \in \mathcal{V}(N)$, define the set of agents who have a larger valuation than their reservation price by

$$U^f(\gamma) \equiv \{j \in N : v_j > f_j\}.$$

Note that $m = \min\{k, |U^f(\gamma)|\}$ objects are allocated under mechanism $\psi^{(f,\triangleright)}$. We define the subset of the m^{th} highest priority agents in set $U^f(\gamma)$ by $U^f_{\triangleright|m}(\gamma)$.

The serial dictatorship mechanism with reservation prices $\psi^{(f,\triangleright)}$ assigns the uniquely determined outcome $\psi^{(f,\triangleright)}(\gamma) \in \mathcal{O}(N, k)$ such that for each $j \in N$,

(a) if $i \in U^f_{\triangleright|m}(\gamma)$, then $\psi_i^{(f,\triangleright)}(\gamma) = (1, f_i)$ and

(b) if $i \notin U^f_{\triangleright|m}(\gamma)$, then $\psi_i^{(f,\triangleright)}(\gamma) = (0, 0)$.

Note that if the reservation prices are zero for all agents, we obtain the classical serial dictatorship mechanism. That is, given the reservation price vector $\mathbf{0} = (0, 0, \ldots) \in \mathcal{F}$ and a priority ordering $\rhd \in \mathcal{P}$, $\psi^{(\mathbf{0},\rhd)}$ is a *serial dictatorship mechanism.*

4 Characterizations

Theorem 1. *A mechanism φ satisfies minimal tradability, individual rationality, strategy-proofness, consistency, independence of unallocated objects, and non wasteful tie-breaking if and only if there exist a reservation price vector $r \in \mathcal{F}$ and a priority ordering $\succ \in \mathcal{P}$ such that φ is a serial dictatorship mechanism with reservation prices based on r and $\succ$, i.e., $\varphi = \psi^{(r,\succ)}$.*

We prove our main result (Theorem 1) in Appendix B. The uniqueness proof proceeds in four parts: first, we construct the individual reservation price vector $r \in \mathcal{F}$; second, we construct the priority ordering $\succ \in \mathcal{P}$ over $\mathbb{N}$; third, we prove that $\varphi = \psi^{(r,\succ)}$ for single object problems, i.e., for $k = 1$; fourth, we extend the result that $\varphi = \psi^{(r,\succ)}$ to any $k \in \mathbb{Z}_+$.

Since the *independence of unallocated objects* is only needed in the last step of the proof of Theorem 1, we obtain the following corollary.

Corollary 1. *For the reduction of our model to single object problems, i.e., for $k = 1$, a mechanism φ satisfies minimal tradability, individual rationality, strategy-proofness, consistency, and non wasteful tie-breaking if and only if there exist a reservation price vector $r \in \mathcal{F}$ and a priority ordering $\succ \in \mathcal{P}$ such that φ is a serial dictatorship mechanism with reservation prices based on r and $\succ$, i.e., $\varphi = \psi^{(r,\succ)}$.*

Theorem 1 and Lemma 1 (b) imply the following corollary.

Corollary 2. *A mechanism φ satisfies minimal tradability, non-imposition, strategy-proofness, consistency, independence of unallocated objects, and non wasteful tie-breaking if and only if there exist a reservation price vector $r \in \mathcal{F}$ and a priority ordering $\succ \in \mathcal{P}$ such that φ is a serial dictatorship mechanism with reservation prices based on r and $\succ$, i.e., $\varphi = \psi^{(r,\succ)}$.*

We prove that each one of the normative properties used in the characterizations in Theorem 1 and Corollaries 1 and 2 is necessary (i.e., we prove the independence of axioms) in Appendix C.

5 An Application

Apart from providing a theoretical foundation for serial dictatorship mechanisms with reservation prices we can provide some insights about how some allocation mechanisms work in various real markets, from the allocation of the next-available consultant-led medical appointment, to on-board flight upgrading, to the prioritization of traffic in a computer network, and so on. For concreteness, we focus on and detail one specific example, the allocation of the next-available consultant-led medical appointment in Australia.

Under Commonwealth federal law, residents in Australia are covered by Medicare universal health insurance, which provides free or subsidized health care services. Private insurance is optional, subscribed to by roughly one in two, and is generally used as a top-up to Medicare, providing additional benefits.[7] For instance, it may reduce or eliminate out-of-pocket costs (also known as "gap payments"). Due to the large number of insurance options and personal circumstances, even for the exact same health care procedure or service, out-of-pocket costs are idiosyncratic.[8]

State and territory governments administer certain elements of health care within their jurisdiction, such as the operation of public hospitals, through charters that set the regulatory framework for the within state provision of medical services. While expressed in plain language, these charters include many requirements that are essentially normative criteria that the service providers should comply with. For example, in the state of Victoria, clinical prioritization requires "equality of access to specialist clinic services" and that specialist clinic appointments are "actively managed to ensure patients are treated equitably within clinically appropriate timeframes and with priority given to patients with an urgent clinical need."[9] In practice, the public hospitals implement these requirements by creating a priority order for specialist services induced by clinical need and arrival time, but which ignores the patients' insurance status. Given the prevailing priority order, the next-available specialist service appointment is offered to the patient with the highest priority, who considers his value and out-of-pocket cost for it, and chooses whether to accept it or not. If the patient accepts, the appointment is allocated to him. Otherwise, the appointment is offered to the patient with the next highest priority, who chooses whether to accept it or not, and so on. All appointments made are nominal and patients cannot trade appointments among themselves.[10]

[7] Private insurance is compulsory for anyone who is not an Australian citizen or a permanent resident.

[8] There are many private insurers, each offering many policies that differ in coverage, "embargo" waiting periods imposed, prices, discounts available, levels of excess or co-payments required, and so on.

[9] See the "Access Policy" white paper by the Health Service Programs Branch (2013).

[10] While our description above applies in many situations, there are several exceptions and limitations.

The *serial dictatorship mechanism with reservation prices* mimics the procedure that the hospitals arrived at for allocating next-available appointments, where the hospital's priority order for specialist services is taken as an exogenously given priority ordering[11] and the out-of-pocket costs to be paid by the patients are interpreted as their reservation price for the service. The patient with the highest priority is considered first. If his value for the next-available appointment is strictly higher than his reservation price, then he obtains this appointment, otherwise, he receives and pays nothing. Either way, we continue with the remaining patients to allocate the next-available appointment.

At the same time, our normative criteria capture some of the requirements set by the states in the charter that specifies the regulatory framework as follows. *Minimal tradability* asks that for any group of patients, there exists a utility profile such that one of them can obtain the appointment. *Individual rationality* specifies that a patient who does not receive an appointment pays nothing, whereas if he receives it, the out-of-pocked amount that he pays cannot exceed his valuation for it.[12] *Strategy-proofness* asks that no patient can profitably misreport his true utility for the appointment; it avoids outcomes that are based on strategic manipulations and levels the playing field, ensuring "equality of access", in the sense that "sophisticated" patients cannot get an edge over "unsophisticated" ones by misreporting their utility. *Consistency* requires that if some patients who did not receive the appointment withdraw from the queue (possibly because they no longer need it) or if some patients take their earlier appointments, the outcome for everyone else remains unchanged. In addition, *consistency* also ensures that the payment of the patient who receives the appointment does not depend on other patients on the waiting list (neither their identities, nor their utilities for the appointment). By *independence of unallocated objects*, unallocated appointment slots do not influence the outcome. *Non wasteful tie-breaking* excludes that an appointment is allocated to a patient at some price if he is indifferent between this outcome and [not receiving an appointment and not paying anything]; thus, an appointment is not "wasted"

For instance, for organ transplants appointments are made using a different dynamic matching procedure (Akbarpour et al., 2018). Emergency room rules for dealing with life threatening situations are also different. More generally, people in very serious conditions are unlikely to pass their turn. Our description is best suited for procedures that are less severe, but nevertheless serious enough to require a specialist-led appointment, including for instance most "watchful waiting" scenarios in which the condition does require a specialist-led appointment that the patients then decide whether or not to take.

[11]Clinical need is most of the time established based on external referrals, the arrival time is random, and hospitals use consistent procedures to determine the priorities. Thus, although the priority order is created within the hospital, it can be interpreted as being essentially exogenously given.

[12]*Non-imposition* allows patients who are not interested in an appointment to withdraw from consideration at no cost.

on a patient who is indifferent since another patient might strictly prefer to receive it.

As a first order approximation, our normative criteria seem to capture reasonably well the requirements set by the states in the charters for the provision of medical services, and our serial dictatorship mechanism with reservation prices mimics the procedure that the hospitals arrived at for the allocation of the next-available specialist-based appointment. In this context, our main characterization result indicates that the current procedure for allocating medical appointments via a serial dictatorship with externally determined reservation prices and priorities is aligned with current public health care guidelines; moreover, this is the only procedure that respects those normative criteria.

6 Conclusion

In a simple setup where homogeneous indivisible objects are allocated to a set of agents, we proposed a set of normative criteria, and we introduced a new *serial dictatorship with reservation prices* mechanism that combines priorities over agents and individual reservation prices. Our main result (Theorem 1) shows that a mechanism φ satisfies *minimal tradability*, *individual rationality*, *strategy-proofness*, *consistency*, *independence of unallocated objects*, and *non wasteful tie-breaking*, if and only if there exists a reservation price vector r and a priority ordering $\succ$ such that φ is a serial dictatorship mechanism based on r and $\succ$.

Our modelling and our results best apply to settings that are similar to the allocation of a consultant-led medical appointment discussed in the previous section. More generally, such settings share a series of similar features. *First*, wealth inequality among agents is common. Consequently, agents may not value money equally and utility comparisons across agents may not be possible, a feature that we capture by representing the preferences of the agents via general utility functions that are not necessarily quasilinear. *Second*, income redistribution is not feasible, a feature that we capture by requiring that in our model agents cannot trade and make transfers among themselves. *Third*, the mechanisms used to generate the outcomes matter, and must satisfy two very specific requirements: first and foremost, procedural fairness, and then compatibility with payments.[13] Our *serial dictatorship with reservation prices* is based on a priority ordering $\succ$, and it is thus procedurally fair, and in addition it is also compatible with the payment vector r.

[13] As we described in Section 5 when considering the allocation of next-available consultant-led medical appointments, the mechanisms are first required to ensure that patients are prioritized based on clinical need and that there is "equality of access". The patients' payments, while important, come second.

While meant to capture the essence of the laws that govern the provision of medical services, some of the properties that we used to characterize our *serial dictatorship mechanisms with reservation prices* are also compatible with other mechanisms, such as for instance *second price auctions*. For a single-object setting, Sakai (2008, Theorem 1) shows that a mechanism satisfies *non-imposition*, *strategy-proofness*, and *efficiency*, if and only if it is a second price auction.[14] In Sakai's model, in contrast to ours, transfers from an agent to another are allowed for and *efficiency* is required, which taken together imply that the object is competitively allocated to the agent who values it the most. Note that unlike in serial dictatorships (*with* or *without* reservation prices), in second price auctions there is no priority ordering to account for. From a normative point of view, this difference is easy to see: *consistency* is key for the construction of priority orderings that underpin serial dictatorships, but a second price auction does not satisfy *consistency*. Thus, contrasting our modelling and results in Corollary 1 with Sakai's yields a clean normative comparison: dropping *consistency* and adding *efficiency* in the presence of *non-imposition* and *strategy-proofness* switches a mechanism from a serial dictatorship with reservation prices to a second price auction.

Appendix

A Proof of Lemma 1

(a) Assume that mechanism φ satisfies *individual rationality*. Let $N \in \mathcal{N}$, $k \in \mathbb{Z}_+$, and $(N, u, k) \in \Gamma(N, k)$ with associated valuation vector $v \in \mathcal{V}(N)$. Let $i \in N$ such that $v_i = 0$. If $\alpha_i(N, u, k) = 0$, then (IR1) implies $\pi_i(N, u, k) = 0$. If $\alpha_i(N, u, k) = 1$, then (IR2) implies $\pi_i(N, u, k) \leq v_i = 0$ and thus, $\pi_i(N, u, k) = 0$. Hence, φ satisfies *non-imposition*.

(b) Assume that mechanism φ satisfies *strategy-proofness* and *non-imposition*. Let $N \in \mathcal{N}$, $k \in \mathbb{Z}_+$, and $(N, u, k) \in \Gamma(N, k)$ with associated valuation vector $v \in \mathcal{V}(N)$. Let $i \in N$ and $u' = (u'_i, u_{-i}) \in \mathcal{U}(N)$ with associated valuation vector $v' = (0, v_{-i}) \in \mathcal{V}(N)$.

(IR1) Suppose that $\alpha_i(N, u, k) = 0$ and, in contradiction to (IR1), $\pi_i(N, u, k) > 0$. By property (i) of utility function u_i, $u_i(\varphi_i(N, u, k)) = u_i(0, \pi_i(N, u, k)) \overset{\text{(i)}}{<} u_i(0, 0)$. By property (ii) of utility function u_i, $u_i(0, 0) \overset{\text{(ii)}}{\leq} u_i(1, 0)$.

By *non-imposition*, we have $\pi_i(N, u', k) = 0$. Hence, $\varphi_i(N, u', k) \in \{(0, 0), (1, 0)\}$ and

[14] Other related characterizations of second price auctions are obtained by Saitoh and Serizawa (2008) and Ohseto (2006).

$u_i(\varphi_i(N,u,k)) < u_i(\varphi_i(N,u',k))$, contradicting *strategy-proofness*. Thus, $\alpha_i(N,u,k) = 0$ implies $\pi_i(N,u,k) = 0$.

(IR2) Suppose that $\alpha_i(N,u,k) = 1$ and, in contradiction to (IR2), $\pi_i(N,u,k) > v_i$ (≥ 0). By property (iii) of utility function u_i, $v_i \neq \infty$ and $u_i(1,v_i) = u_i(0,0)$. By property (i) of utility function u_i, $u_i(\varphi_i(N,u,k)) = u_i(1,\pi_i(N,u,k)) \overset{\text{(i)}}{<} u_i(1,v_i) = u_i(0,0)$. By property (ii) of utility function u_i, $u_i(0,0) \overset{\text{(ii)}}{\leq} u_i(1,0)$.

By *non-imposition*, we have $\pi_i(N,u',k) = 0$. Hence, $\varphi_i(N,u',k) \in \{(0,0),(1,0)\}$ and $u_i(\varphi_i(N,u,k)) < u_i(\varphi_i(N,u',k))$, contradicting *strategy-proofness*. Thus, $\alpha_i(N,u,k) = 1$ implies $\pi_i(N,u,k) \leq v_i$. $\square$

B Proof of Theorem 1

It is easy to see that any serial dictatorship mechanism with reservation prices induced by some reservation price vector $f \in \mathcal{F}$ and some priority ordering $\succ \in \mathcal{P}$ satisfies all the properties in the theorem.

For the uniqueness proof we assume that φ satisfies all the properties in the theorem; we split the proof into four parts: first, we construct the individual reservation price vector $r \in \mathcal{F}$; second, we construct the priority ordering $\succ \in \mathcal{P}$ over $\mathbb{N}$; third, we prove that $\varphi = \psi^{(r,\succ)}$ for single object problems, i.e., for $k = 1$; fourth, we extend the result that $\varphi = \psi^{(r,\succ)}$ to any $k \in \mathbb{Z}_+$ via an induction argument.

Part 1: Individual Reservation Prices

We first establish the existence of an individual reservation price vector.

Lemma 2. *Assume that mechanism φ satisfies minimal tradability, individual rationality, and strategy-proofness. Then, for each agent $i \in \mathbb{N}$, there exists an individual reservation price $r_i \geq 0$ such that for each utility function $u_i \in \mathcal{U}(\{i\})$ with associated valuation $v_i \in \mathcal{V}(\{i\})$:*

(i) *$v_i > r_i$ implies $\varphi_i(\{i\},u_i,1) = (1,r_i)$,*

(ii) *$v_i = r_i$ implies $\varphi_i(\{i\},u_i,1) \in \{(0,0),(1,r_i)\}$, and*

(iii) *$v_i < r_i$ implies $\varphi_i(\{i\},u_i,1) = (0,0)$.*

Proof. For each $i \in \mathbb{N}$, we define an *individual reservation price* $r_i \geq 0$ as follows. Let $N = \{i\}$ and $k = 1$. Define the *price range* of mechanism φ for agent i with preferences u_i as the set of all possible prices at which he could obtain the object, i.e.,

$$P_i^{\varphi} = \{p_i \in \mathbb{R}_+ : \varphi_i(\{i\}, u_i, 1) = (1, p_i) \text{ for some } u_i \in \mathcal{U}(\{i\})\}.$$

We show that $|P_i|^{\varphi} \leq 1$.

Suppose that $|P_i^{\varphi}| > 1$. Then, there exist $p_i, p_i' \in P_i^{\varphi}$ and, without loss of generality, assume $p_i > p_i'$. Hence, there exist utility functions $u_i, u_i' \in \mathcal{U}(\{i\})$ such that $\varphi_i(\{i\}, u_i, 1) = (1, p_i)$ and $\varphi_i(\{i\}, u_i', 1) = (1, p_i')$. Then, agent i with preferences represented by u_i can receive the object at the lower price p_i' if he pretends his preferences are represented by u_i'. Thus, in contradiction to *strategy-proofness*, by property (i) of utility function u_i, $u_i(\varphi_i(\{i\}, u_i', 1)) = u_i(1, p_i') \overset{\text{(i)}}{>} u_i(1, p_i) = u_i(\varphi_i(\{i\}, u_i, 1))$.

If $P_i^{\varphi} \equiv \emptyset$, then we set $r_i = \infty$. Otherwise, r_i is defined via $P_i^{\varphi} = \{r_i\}$.

Recall that by definition, for each $i \in \mathbb{N}$, we have that for the valuation v_i associated with u_i, $v_i \geq 0$. For each $i \in \mathbb{N}$, *minimal tradability* implies that $r_i < \infty$ and *individual rationality* (IR2) implies that $v_i \geq r_i$. Note that at $v_i = r_i < \infty$, agent i is indifferent between [not receiving the object and not paying anything] and [receiving the object and paying $r_i = v_i$]. $\square$

With our next lemma, we show that for any problem, if an agent receives an object, then his valuation has to be weakly larger than his individual reservation price (which also equals his payment); otherwise, his payment is necessarily null.

Lemma 3. *Assume that mechanism φ satisfies minimal tradability, individual rationality, strategy-proofness, consistency, and independence of unallocated objects. Then, for each $N \in \mathcal{N}$, each $k \in \mathbb{Z}_+$, each $\gamma \in \Gamma(N, k)$ with associated valuation vector $v \in \mathcal{V}(N)$, and each $i \in N$, if $\alpha_i(\gamma) = 1$, then $\pi_i(\gamma) = r_i \leq v_i$ (with r_i as in Lemma 2). Furthermore, if $\gamma = (N, u, 1)$, i.e., $k = 1$, then independence of unallocated objects is not necessary.*

Proof. Assume that mechanism φ satisfies all the properties in the lemma. Let $N \in \mathcal{N}$, $k \in \mathbb{Z}_+$, and $\gamma = (N, u, k) \in \Gamma(N, k)$ with associated valuation vector $v \in \mathcal{V}(N)$. Let $i \in N$ and $\alpha_i(\gamma) = 1$. If all agents but agent i leave with their allotments, then the reduced problem is $\gamma_{\{i\}} = (\{i\}, u_i, k_{\{i\}})$ where $k_{\{i\}} = k - \sum_{j \in N \setminus \{i\}} \alpha_j(\gamma)$. By *consistency*, $\varphi_i(\gamma_{\{i\}}) = \varphi_i(\gamma)$ and $\alpha_i(\gamma_{\{i\}}) = \alpha_i(\gamma) = 1$. If $k_{\{i\}} = 1$, then $\gamma_{\{i\}} = (\{i\}, u_i, 1)$. If $k_{\{i\}} > 1$, then using *independence of unallocated objects*, we obtain $\varphi_i(\{i\}, u_i, 1) = \varphi_i(\gamma_{\{i\}})$.

Thus, $\varphi_i(\{i\}, u_i, 1) = \varphi_i(\gamma)$ and $\alpha_i(\{i\}, u_i, 1) = \alpha_i(\gamma) = 1$. By Lemma 2, $v_i \geq r_i$ and $\varphi_i(\{i\}, u_i, 1) = (1, r_i) = \varphi_i(\gamma)$. In particular, $\pi_i(\gamma) = r_i \leq v_i$.

If $k = 1$, then in the proof above, $k_{\{i\}} = 1$ and *independence of unallocated objects* is not necessary. □

Part 2: Priority Ordering

In this part, we consider single object problems, i.e., $k = 1$.

Let $i, j \in \mathbb{N}$, $i \neq j$. By *minimal tradability*, there exists $u = (u_i, u_j) \in \mathcal{U}(\{i, j\})$ with associated valuation vector $v = (v_x, v_y) \in \mathcal{V}(\{x, y\})$ such that for $x \in \{i, j\} \equiv \{x, y\}$, $\alpha_x(\{x, y\}, u, 1) = 1$. By *consistency* and Lemma 2 (i) and (ii), $\varphi_x(\{x, y\}, u, 1) = \varphi_x(\{x\}, u_x, 1) = (1, r_x)$.

Let $u' = (\bar{u}_x, u_y) \in \mathcal{U}(\{x, y\})$ with associated valuation vector $v' = (r_x + 1, v_y) \in \mathcal{V}(\{x, y\})$. Then, by *strategy-proofness*, $\alpha_x(\{x, y\}, u', 1) = 1$ (in fact, we even have $\varphi_x(\{x, y\}, u', 1) = (1, r_x)$).

Let $(\bar{u}_x, \bar{u}_y) \in \mathcal{U}(\{x, y\})$ with associated valuation vector $(r_x + 1, r_y + 1) \in \mathcal{V}(\{x, y\})$. By *consistency*, the object continues to remain allocated at problem $(\{x, y\}, (\bar{u}_x, \bar{u}_y), 1)$. To see this, observe that otherwise, if the object is not allocated anymore, starting from $(\{x, y\}, (\bar{u}_x, \bar{u}_y), 1)$ and removing agent y, by *consistency* we would have $\alpha_x(\{x\}, \bar{u}_x, 1) = 0$, which would contradict that $\varphi_x(\{x\}, \bar{u}_x, 1) = (1, r_x)$ (by Lemma 2 (i)). Thus, one of the agents in $\{i, j\} \equiv \{x, y\}$ receives the object. If $\alpha_i(\{i, j\}, (\bar{u}_i, \bar{u}_j), 1) = 1$, then set $i \succ j$. Otherwise, if $\alpha_j(\{i, j\}, (\bar{u}_i, \bar{u}_j), 1) = 1$, then set $j \succ i$.

We now prove the transitivity of $\succ$. Assume, by contradiction, that there exist distinct agents $i, j, l \in \mathbb{N}$ such that $i \succ j$, $j \succ l$, and $l \succ i$. Assume that for any of these agents $a \in \{i, j, l\}$, $\bar{u}_a$ is the utility function used to determine $\succ$ with associated valuation $r_a + 1$. Hence, $\alpha_i(\{i, j\}, (\bar{u}_i, \bar{u}_j), 1) = 1$, $\alpha_j(\{j, l\}, (\bar{u}_j, \bar{u}_l), 1) = 1$, and $\alpha_l(\{i, l\}, (\bar{u}_i, \bar{u}_l), 1) = 1$.

By *minimal tradability*, there exists $u = (u_i, u_j, u_l) \in \mathcal{U}(\{i, j, l\})$ with associated valuation vector $v = (v_x, v_y, v_z) \in \mathcal{V}(\{x, y, z\})$ such that for $x \in \{i, j, l\} \equiv \{x, y, z\}$, $\alpha_x(\{x, y, z\}, u, 1) = 1$. By *consistency* and Lemma 2 (i) and (ii), $\varphi_x(\{x, y, z\}, u, 1) = \varphi_x(\{x\}, u_x, 1) = (1, r_x)$.

Let $u' = (\bar{u}_x, u_y, u_z) \in \mathcal{U}(\{x, y, z\})$ with associated valuation vector $v' = (r_x + 1, v_y, v_z) \in \mathcal{V}(\{x, y, z\})$. Then, by *strategy-proofness*, $\alpha_x(\{x, y, z\}, u', 1) = 1$ (in fact, we even have $\varphi_x(\{x, y, z\}, u', 1) = (1, r_x)$).

Let $u'' = (\bar{u}_x, \bar{u}_y, u_z) \in \mathcal{U}(\{x, y, z\})$ with associated valuation vector $v'' = (r_x + 1, r_y + 1, v_z) \in \mathcal{V}(\{x, y, z\})$. By *consistency*, the object continues to remain allocated at problem

$(\{x,y,z\},u'',1)$. To see this, observe that otherwise, if the object is not allocated anymore, starting from $(\{x,y,z\},u'',1)$ and removing agent z, by *consistency* we would have $\alpha_x(\{x,y\},(\bar{u}_x,\bar{u}_y),1)=0$ and $\alpha_y(\{x,y\},(\bar{u}_x,\bar{u}_y,1))=0$, which would contradict that either $x\succ y$ or $y\succ x$. Thus, one of the agents in $\{i,j,l\}\equiv\{x,y,z\}$ receives the object.

Let $(\bar{u}_x,\bar{u}_y,\bar{u}_z)\in\mathcal{U}(\{x,y,z\})$ with associated valuation vector $(r_x+1,r_y+1,r_z+1)\in\mathcal{V}(\{x,y,z\})$. By *consistency* (if agent z did not receive the object before) or by *strategy-proofness* (if agent z did receive the object before), one of the agents in $\{i,j,l\}\equiv\{x,y,z\}$ receives the object, without loss of generality, agent i, i.e., $\alpha_i(\{i,j,l\},(\bar{u}_i,\bar{u}_j,\bar{u}_l),1)=1$. By *consistency*, $\alpha_i(\{i,l\},(\bar{u}_i,\bar{u}_l),1)=1$, contradicting $l\succ i$ (and hence, $\alpha_l(\{i,l\},(\bar{u}_i,\bar{u}_l),1)=1$).

Part 3: Single Object Problems

We show for single object problems, i.e., $k=1$, that φ always assigns the object and payments as if it is a serial dictatorship mechanism based on $r\in\mathcal{F}$ and $\succ\in\mathcal{P}$. That is, we show that for each $N\in\mathcal{N}$, each $\gamma=(N,u,1)\in\Gamma(N,1)$ with associated valuation vector $v\in\mathcal{V}(N)$, and $U^r(\gamma)\equiv\{j\in N: v_j>r_j\}$, φ assigns the uniquely determined outcome such that for each $i\in N$,

(a) if $i=\arg\max_{\succ}U^r(\gamma)$, then $\psi_i^{(r,\succ)}(\gamma)=(1,r_i)$ and

(b) if $i\neq\arg\max_{\succ}U^r(\gamma)$, then $\psi_i^{(r,\succ)}(\gamma)=(0,0)$.

Recall that by *individual rationality* (IR1), if $i\in N$ and $\alpha_i(\gamma)=0$, then $\pi_i(\gamma)=0$. Furthermore, by Lemma 3, if $i\in N$ and $\alpha_i(\gamma)=1$, then $\pi_i(\gamma)=r_i$. Hence, we only need to prove that the allocation rule $\alpha=\alpha^{(r,\succ)}$. We proceed by contradiction, considering a different object allocation in each of the Cases (a) and (b); to simplify the proof, we start with Case (b).

Case (b): there exists $i\neq\arg\max_{\succ}U^r(\gamma)$ such that $\alpha_i(\gamma)=1$.

Case (b.1): $i\notin U^r(\gamma)$

By Lemma 3, $\pi_i(\gamma)=r_i$ and by $i\notin U^r(\gamma)$, we have $v_i\leq r_i$. If $v_i<\pi_i(\gamma)$, then *individual rationality* (IR2) is violated. If $v_i=\pi_i(\gamma)$, then *non wasteful tie-breaking* is violated.

Case (b.2): $i\in U^r(\gamma)$ but there exists an agent $j\in U^r(\gamma)$ such that $j\succ i$ and $\alpha_i(\gamma)=1$.

Assume that $(\bar{u}_i,\bar{u}_j)$ is the utility profile used to determine $j\succ i$ with associated valuation vector (r_i+1,r_j+1). Hence, $\varphi_j(\{i,j\},(\bar{u}_i,\bar{u}_j),1)=(1,r_j)$.

Starting from problem $(N, u, 1)$, by *consistency* and *Lemma 2*, $\varphi_i(\{i,j\},(u_i,u_j),1) = (1, r_i)$. By *strategy-proofness*, $\alpha_i(\{i,j\},(\bar{u}_i,u_j),1) = 1$. Hence, $\alpha_j(\{i,j\},(\bar{u}_i,u_j),1) = 0$ and by *individual rationality* (IR1), $\varphi_j(\{i,j\},(\bar{u}_i,u_j),1) = (0,0)$.

Since $j \in U^r(\gamma)$, we have $v_j > r_j$. Then, in contradiction to *strategy-proofness*, we have that $u_j(\varphi_j(\{i,j\},(\bar{u}_i,\bar{u}_j),1)) = u_j(1,r_j) > u_j(0,0) = u_j(\varphi_j(\{i,j\},(\bar{u}_i,u_j),1))$ (agent j with utility function u_j and valuation v_j will beneficially misreport utility function $\bar{u}_j$ with valuation $r_j + 1$).

Case (a): for $j = \arg\max_{\succ} U^r(\gamma)$, we have $\alpha_j(\gamma) = 0$.

The contradiction obtained for Case (b) above implies that for each $i \in N \setminus \{j\}$, $\alpha_i(\gamma) = 0$. If now also $\alpha_j(\gamma) = 0$, then the object is not allocated. By *consistency*, starting from problem $\gamma = (N, u, 1)$ and removing all agents but j, we obtain $\alpha_j(\{j\}, u_j, 1) = 0$. However, since $j \in U^r(\gamma)$, we have $v_j > r_j$, which by the definition of r_j in Lemma 2 (i) implies that $\alpha_j(\{j\}, u_j, 1) = 1$, a contradiction. □

Part 4: An Arbitrary Number of Objects

We now show by induction on the number of objects that $\varphi = \psi^{(r,\succ)}$ for the general domain of all problems, i.e., $k \in \mathbb{Z}_+$.

Induction Basis $\boldsymbol{k = 0, 1}$: Let $N \in \mathcal{N}$, $k \in \{0,1\}$, and $\gamma = (N, u, k) \in \Gamma(N, k)$. Then, $\varphi(\gamma) = \psi^{(r,\succ)}(\gamma)$ follows for $k = 0$ by *individual rationality* (IR1) and for $k = 1$ by Part 3.

Induction Hypothesis $\boldsymbol{k' \leq k}$: On the subdomain of problems where at most $k \geq 1$ objects are available, we assume $\varphi = \psi^{(r,\succ)}$.

Induction Step $\boldsymbol{k + 1}$: We show that for problems where $k + 1$ objects are available, we have $\varphi = \psi^{(r,\succ)}$.

Suppose, by contradiction, that there exists a set of agents $N \in \mathcal{N}$ and a utility profile $u \in \mathcal{U}(N)$ such that

$$\varphi(N, u, k+1) \neq \psi^{(r,\succ)}(N, u, k+1).$$

Hence, there exists an agent $i \in N$ such that $\varphi_i(N, u, k+1) \neq \psi_i^{(r,\succ)}(N, u, k+1)$. Let

$$\varphi(N, u, k+1) = (\alpha(N, u, k+1), \pi(N, u, k+1))$$

and

$$\psi^{(r,\succ)}(N, u, k+1) = (\alpha'(N, u, k+1), \pi'(N, u, k+1)).$$

Recall that by *individual rationality* (IR1), if $\alpha_i(\gamma) = 0$, then $\pi_i(\gamma) = 0$ and by Lemma 3, if $\alpha_i(\gamma) = 1$, then $\pi_i(\gamma) = r_i$. Hence,

$$\alpha_i(N, u, k+1) \neq \alpha'_i(N, u, k+1). \tag{1}$$

Without loss of generality, assume $\alpha_i(N, u, k+1) = 1$ and $\alpha'_i(N, u, k+1) = 0$.

Any remaining agent $j \in N \setminus \{i\}$ is of one of the following four types.

	type 1	type 2	type 3	type 4
$\alpha_j(N, u, k+1)$	0	1	0	1
$\alpha'_j(N, u, k+1)$	0	0	1	1

First, assume an agent $j \in N \setminus \{i\}$ is of type 4. When agent j leaves with his allotment, under both mechanisms φ and $\psi^{(r,\succ)}$, we obtain the reduced problem $(N \setminus \{j\}, u_{N\setminus\{j\}}, k)$. By *consistency*, we then have

$$\varphi_i(N, u, k+1) = \varphi_i(N \setminus \{j\}, u_{N\setminus\{j\}}, k)$$

and

$$\psi_i^{(r,\succ)}(N, u, k+1) = \psi_i^{(r,\succ)}(N \setminus \{j\}, u_{N\setminus\{j\}}, k).$$

By inequality (1),

$$\varphi_i(N \setminus \{j\}, u_{N\setminus\{j\}}, k) \neq \psi_i^{(r,\succ)}(N \setminus \{j\}, u_{N\setminus\{j\}}, k),$$

contradicting the Induction Hypothesis.

Second, assume an agent $j \in N \setminus \{i\}$ is of type 3. When all agents except agents i and j leave with their allotments, under mechanism φ, we obtain the reduced problem $(\{i,j\}, u_{\{i,j\}}, k')$ where $1 \leq k' \leq k+1$ and under mechanism $\psi^{(r,\succ)}$, we obtain the reduced problem $(\{i,j\}, u_{\{i,j\}}, k'')$ where $1 \leq k'' \leq k+1$. By *consistency*, we then have

$$\varphi_i(N, u, k+1) = \varphi_i(\{i,j\}, u_{\{i,j\}}, k')$$

and

$$\psi_i^{(r,\succ)}(N, u, k+1) = \psi_i^{(r,\succ)}(\{i,j\}, u_{\{i,j\}}, k'').$$

Note that in each reduced problem exactly one object is allocated but the total numbers of unallocated objects $k'-1$ and $k''-1$ might differ. When removing all unallocated objects from both reduced problems, under both mechanisms φ and $\psi^{(r,\succ)}$, we obtain the problem $(\{i,j\}, u_{\{i,j\}}, 1)$. By *independence of unallocated objects*, we then have

$$\varphi_i(N,u,k+1) = \varphi_i(\{i,j\}, u_{\{i,j\}}, k') = \varphi_i(\{i,j\}, u_{\{i,j\}}, 1)$$

and

$$\psi_i^{(r,\succ)}(N,u,k+1) = \psi_i^{(r,\succ)}(\{i,j\}, u_{\{i,j\}}, k'') = \psi_i^{(r,\succ)}(\{i,j\}, u_{\{i,j\}}, 1).$$

By inequality (1),

$$\varphi_i(\{i,j\}, u_{\{i,j\}}, 1) \neq \psi_i^{(r,\succ)}(\{i,j\}, u_{\{i,j\}}, 1),$$

contradicting the Induction Basis.

Hence, by the previous two arguments, an agent $j \in N \setminus \{i\}$ cannot be of type 3 or 4. Thus, for each agent $j \in N \setminus \{i\}$ we have

$$\alpha'_j(N,u,k+1) = 0.$$

Consider mechanism φ. When all agents except agent i leave with their allotments, we obtain the reduced problem $(\{i\}, u_i, k''')$ where $1 \leq k''' \leq k+1$. By *consistency*, we then have

$$\varphi_i(N,u,k+1) = \varphi_i(\{i\}, u_i, k''').$$

When removing all unallocated objects from the reduced problem $(\{i\}, u_i, k''')$, we obtain the problem $(\{i\}, u_i, 1)$. By *independence of unallocated objects*, we then have

$$\varphi_i(N,u,k+1) = \varphi_i(\{i\}, u_i, k''') = \varphi_i(\{i\}, u_i, 1).$$

Thus, we have

$$\alpha_i(N,u,k+1) = \alpha_i(\{i\}, u_i, k''') = \alpha_i(\{i\}, u_i, 1) = 1.$$

Since $\alpha_i(\{i\}, u_i, 1) = 1$, by Lemma 3, agent i's valuation $v_i \geq r_i$. Then, by *non wasteful tie-breaking*, we have

$$v_i > r_i. \tag{2}$$

Consider mechanism $\psi^{(r,\succ)}$ (but note that we will only use $\psi^{(r,\succ)}$'s properties but not its definition). When all agents except agents i leave with their allotments, we obtain the reduced problem $(\{i\}, u_i, k+1)$ (we know that $k+1$ objects remain because all leaving agents are of type 1 or 2). By *consistency*, we then have

$$\psi_i^{(r,\succ)}(N, u, k+1) = \psi_i^{(r,\succ)}(\{i\}, u_i, k+1),$$

and in particular,

$$\alpha_i'(N, u, k+1) = \alpha_i'(\{i\}, u_i, k+1) = 0.$$

By *individual rationality* (IR1), $\pi_i'(\{i\}, u_i, k+1) = 0$ and

$$\psi_i^{(r,\succ)}(\{i\}, u_i, k+1) = (0, 0). \tag{3}$$

By *minimal tradability* there exists a utility function $\hat{u}_i$ such that $\alpha'(\{i\}, \hat{u}_i, k+1) = 1$. When removing all unallocated objects from the reduced problem $(\{i\}, \hat{u}_i, k+1)$, we obtain the problem $(\{i\}, \hat{u}_i, 1)$. By *independence of unallocated objects*, we then have

$$\psi_i^{(r,\succ)}(\{i\}, \hat{u}_i, k+1) = \psi_i^{(r,\succ)}(\{i\}, \hat{u}_i, 1),$$

and in particular,

$$\alpha_i'(\{i\}, \hat{u}_i, k+1) = \alpha_i'(\{i\}, \hat{u}_i, 1) = 1.$$

By Lemma 3, $\pi_i'(\{i\}, \hat{u}_i, 1) = r_i$ and

$$\psi_i^{(r,\succ)}(\{i\}, \hat{u}_i, k+1) = \psi_i^{(r,\succ)}(\{i\}, \hat{u}_i, 1) = (1, r_i). \tag{4}$$

By (in)equalities (2), (3), (4), and properties (i) and (ii) of utility function u_i, we obtain

$$u_i\left(\psi_i^{(r,\succ)}(\{i\}, \hat{u}_i, k+1)\right) \overset{(4)}{=} u_i(1, r_i) \overset{(2)\&(\text{i})}{>} u_i(1, v_i) \overset{(\text{ii})}{=} u_i(0, 0) \overset{(3)}{=} u_i\left(\psi_i^{(r,\succ)}(\{i\}, u_i, k+1)\right),$$

contradicting *strategy-proofness* (agent i with utility function u_i and valuation v_i will beneficially misreport utility function $\hat{u}_i$).

Now it follows that inequality (1) cannot hold. Hence, $\alpha_i(N, u, k+1) = \alpha_i'(N, u, k+1)$ and $\varphi_i(N, u, k+1) = \psi_i^{(r,\succ)}(N, u, k+1)$. $\square$

C Independence of Axioms

The following examples present mechanisms that satisfy all the properties in Theorem 1 and Corollary 2, except for the one(s) in the title of the example.

Example 1 (***Minimal Tradability***)
The *no-trade mechanism* never allocates any object and no payments are made.

Example 2 (***Individual Rationality, Non-Imposition***)
Fix a positive price $P > 0$ and assign objects sequentially at price $P > 0$ to the agents with the lowest indices in N, until we run out of objects or agents, all remaining agents pay nothing.

This mechanism, φ^1, does neither satisfy *individual rationality* (IR2) nor *non-imposition*, e.g., for problem $\gamma = (N, u, 1)$ with $1 \in N$ and u_1 such that $v_1 = 0$, we have $\alpha_1^1(\gamma) = 1$ and $\pi_1^1(\gamma) = P > 0 = v_1$. However, φ^1 satisfies *individual rationality* (IR1).

Alternatively, fix a positive price $P > 0$ and assign objects sequentially at price $P > 0$ to the agents with the lowest indices within the set of agents who have a valuation larger than P, until we run out of objects or agents, all remaining agents, except agent 1, pay nothing; if agent 1 is present, even if his valuation is not larger than P, then he pays price P.

This mechanism, φ^2, does neither satisfy *individual rationality* (IR1) nor *non-imposition*, e.g., for problem $\gamma = (N, u, 1)$ with $1 \in N$ and u_1 such that $v_1 = 0$, we have $\alpha_1^2(\gamma) = 0$ and $\pi_1^2(\gamma) = P > 0$. However, φ^2 satisfies *individual rationality* (IR2).

Example 3 (***Strategy-Proofness***)
We assign objects sequentially to the agents with the lowest indices within the set of agents who have a positive valuation, until we run out of objects or agents, agents who obtain an object pay half their valuation, all remaining agents pay nothing.

Example 4 (***Consistency***)
Let $f \in \mathcal{F}$ be a vector of reservation prices and $\rhd, \rhd' \in \mathcal{P}$ be two distinct priority orderings. We apply $\psi^{(f,\rhd)}$ to problems $\gamma \in \Gamma(N, k)$ where the set of agents N has cardinality 2 and $\psi^{(f,\rhd')}$ otherwise.

Example 5 (***Independence of Unallocated Objects***)
Mechanism φ' is defined as follows. Let $f \in \mathcal{F}$ be a vector of reservation prices and $\rhd \in \mathcal{P}$ be a priority ordering. Then, if fewer agents than there are objects want an object (i.e., their valuation is higher than their reservation price), no object is allocated and no payment is made, i.e., φ' coincides with the no trade mechanism. Otherwise, $\varphi' = \psi^{(f,\rhd)}$.

Note that this is an adjustment of the no trade mechanism in such a way that if at least as many agents as there are objects want an object, all objects are allocated and hence *minimal tradability* is satisfied. Furthermore, for single object problems, we have $\varphi' = \psi^{(f,\triangleright)}$. For problems with $k > 1$, the cases (i) "fewer agents than there are objects want an object" and (ii) "at least as many agents than there are objects want an object" are unchanged when agents leave with their allotments, and hence *consistency* is satisfied.

Example 6 (***Non Wasteful Tie-Breaking***)
Consider a modification of our serial dictatorship mechanism with reservation prices in which also agents who are indifferent between [not receiving the object and not paying anything] and [receiving the object and paying his reservation price], as long as objects are still available, receive an object.

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