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Comparative statics for size-dependent discounts in matching markets☆

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ABSTRACT

We prove a natural comparative static for many-to-many matching markets in which agents' choice functions exhibit *size-dependent discounts*: reducing the extent to which some agent discounts additional partners leads to improved outcomes for the agents on the other side of the market, and worsened outcomes for the agents on the same side of the market. Our argument draws upon recently developed methods bringing tools from choice theory into matching.

1. Introduction

Delacrétaz et al. (2019) introduced a general family of valuation functions with *size-dependent discounts*, under which each agent's value for different sets of potential partners is given by an additive valuation over individual agents, discounted by a term that depends on the total number of agents in the set.[4] The additivity of a's valuation ensures that each agent with whom a may partner is evaluated independently from any other agent in the set; thus, there are *no externalities* among potential partners. Meanwhile, the discount term captures the idea that agent a's marginal value for partners decreases as his number of partners increases; thus, there is *competition* among potential partners.

We study many-to-many matching markets with size-dependent discounts and investigate the welfare implications of a discount reduction, under which an agent becomes more willing to accept additional trading partners. Our main result (Theorem 1) shows the intuitive comparative static that a decrease in one agent's discounts makes all other agents on his side of the market worse off, and all agents on the other side of the market better off. To prove our main result, we draw upon recently developed methods that bring tools from choice theory into matching: First, we show that valuation functions with size-dependent discounts induce *path-independent* choice functions; moreover, discount reductions lead to *expansions* of those choice functions. Then, to complete our proof, we invoke a powerful comparative static result of Chambers and Yenmez (2017) that applies to all expansions of path-independent choice functions.[5] We also show that our main conclusion continues to hold if the discounts of *all* agents on one side of the market decrease (Corollary 1). However, the effect of a simultaneous reduction in the discounts of agents on *opposite* sides of the market is ambiguous, even if the discounts decrease by the exact same amount (Example 1). Finally, we

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4 Relatedly, Farrell and Scotchmer (1988) modeled a situation in which the productivity of a group of agents depends on the number of agents in the group but not on their identities.

5 As we discuss at the end of Section 3, for an alternative way to complete our proof, we could use the comparative static result of Pycia and Yenmez (2019) instead of that of Chambers and Yenmez (2017), as the two results are equivalent in our context.

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show that our main findings can be sharpened to cover deferred-acceptance-like mechanisms by comparing side-optimal stable matchings (Corollary 2).

2. Model

7There are finite sets of *workers* W and *firms* F. The full set of *agents* is I, where $I = W \cup F$ (and $W \cap F = \emptyset$). We denote by $|A|$ the number of agents in $A \subseteq I$, and write $\mathcal{P}(I) = 2^I$ for the set of all subsets of agents.

For each agent $a \in I$, worker or firm, we let

$$\bar{I}^a \equiv \begin{cases} F & a \in W \\ W & a \in F \end{cases}$$

denote the set of agents on the *other side* of the market from a; we refer to the set of agents $I^a = I \setminus \bar{I}^a$ as being on the *same side* of the market as a.

For each agent $a \in I$, the *valuation* function $v^a : \bar{I}^a \to \mathbb{R}$ assigns a value to each individual agent on the other side of the market. We assume that each agent a can withdraw from the market at some cost, which for simplicity we normalize to 0, i.e., $v^a(\emptyset) = 0$.

We extend each agent a's valuation over individual agents to sets of agents by assuming *size-dependent discounts*: we require that there exists a vector of *discounts* $\delta^a = (\delta^a_1, \ldots, \delta^a_j, \ldots, \delta^a_{|\bar{I}^a|})$, with $\delta^a_1 = 0$, $\delta^a_{j-1} \le \delta^a_j$ for each $1 < j \le |\bar{I}^a|$, and such that for each $A \subseteq \bar{I}^a$,[6]

$$v^a_\delta(A) = \sum_{b \in A} v^a(b) - \sum_{j=1}^{|A|} \delta^a_j. \tag{1}$$

We assume that each agent's valuation and discounts are such that the induced preferences over sets of agents on the other side of the market are strict.[7]

The lists of size-dependent discounts δ^a and valuations v^a_δ for each agent a give rise to the *discount profile* $\delta = (\delta^a)_{a \in I}$ and associated *valuation profile* $v^I_\delta = (v^a_\delta)_{a \in I}$.

For each agent $a \in I$, the *choice function* induced by a's valuation selects that agent's most preferred set of other-side agents from any given $A \subseteq \bar{I}^a$, i.e.,

$$C^a_\delta(A) = \arg\max_{A' \subseteq A} \{v^a_\delta(A')\}.$$

Note that since agents' induced preferences over sets of partners are strict, we always have $|C^a_\delta(A)| = 1$, and thus we may think of $C^a_\delta(A)$ as a set of agents.

A *(many-to-many) matching* μ assigns to each agent a a set of agents on the other side of the market $\bar{I}^a$, allowing for the possibility that a remains unmatched. That is, $\mu(a) \in \mathcal{P}(\bar{I}^a)$ and we have $b \in \mu(a)$ if and only if $a \in \mu(b)$. A matching μ is *(revealed) preferred* to a matching μ' by agent a if

$$C^a_\delta(\mu(a) \cup \mu'(a)) = \mu(a).$$

A matching μ is *individually rational* if no agent a prefers to unilaterally drop some of the agents with whom he is matched at $\mu(a)$, i.e., if for each $a \in I$, we have $C^a_\delta(\mu(a)) = \mu(a)$. A matching μ is *unblocked* if there does not exist a nonempty set of worker–firm pairs $X \subseteq W \times F$ such that, for every $a \in I$, letting $\overline{X}^a \equiv \{b \in \bar{I}^a : \{a, b\} \in X\}$, we have $\overline{X}^a \cap \mu(a) = \emptyset$ and $\overline{X}^a \subseteq C^a_\delta(\mu(a) \cup \overline{X}^a)$. A matching μ is *stable* if it is both individually rational and unblocked.

6 For simplicity, with some abuse of notation, we write v^a_δ instead of $v^a_{\delta^a}$.

7 Our size-dependent discounts concept corresponds to the "one block" non-monotonic version of Delacrétaz et al. (2019).

2.1. Illustration

Valuations with size-dependent discounts naturally expand the class of preferences that agents are usually allowed to express in two-sided matching markets with capacity constraints.[8] Indeed, in prior two-sided models of matching with capacity constraints, the constraints are typically "hard", in the sense that they are fixed irrespective of which partners are available.[9] Moreover, those prior models often assume a coarse form for the language an agent may use to express his preferences over partners—such as requiring a single rank-order list (with the resulting preferences being *responsive*; see Roth, 1985). Under size-dependent discount valuations, by contrast, the number of available positions (and associated cut-offs) can be more flexible.

For instance, in a school choice setting, we might think of a school as being able to accommodate anywhere between $N - \underline{n}$ and $N + \bar{n}$ students (with $\underline{n}$ not necessarily equal to $\bar{n}$), where the number of admitted students is more finely calibrated via a series of increasing size-dependent discount factors

$$\delta_{N-\underline{n}} \le \cdots \le \delta_N \le \cdots \le \delta_{N+\bar{n}}.$$

The interpretation is that the school's maximal physical capacity is $N + \bar{n}$; and "on average" a group of N students should be admitted, but only if all of their application scores are above δ_N; moreover, the number of available positions flexes upwards and downwards as a function of the quality of the applicants.[10] Note that the ability to express "soft" capacity constraints may help open up the possibility of achieving *fair* and *non-wasteful* assignments of students to schools (Ehlers et al., 2014).[11]

3. Results

Our main result shows that under stable matching, reducing some agent's discounts leads to worse outcomes for other agents on the same side of the market, and better outcomes for agents on the other side of the market. Formally, for an agent a with discounts δ^a, a *discount reduction* occurs when we decrease some subset of the agent's discount values, i.e., we replace a's discounts δ^a with $\varepsilon^a \equiv (\varepsilon^a_1, \ldots, \varepsilon^a_k, \ldots, \varepsilon^a_{|\bar{I}^a|})$ such that $\varepsilon^a_k \le \delta^a_k$ for each $1 \le k \le |\bar{I}^a|$ (and $\varepsilon^a_1 = \delta^a_1 = 0$).[12] We use the standard notation $v^{-a}_\delta = v^{I \setminus \{a\}}_\delta$, and denote by $(v^a_\varepsilon, v^{-a}_\delta)$ the "post-discount-reduction" valuation profile obtained from v^I_δ by replacing v^a_δ with v^a_ε.

Our main result is then as follows.

Theorem 1. *Consider any profile of size-dependent discounts δ with associated valuation profile v^I_δ. Fix any agent $a \in I$, and consider any discount reduction from δ^a to ε^a, with associated post-discount-reduction valuation profile $(v^a_\varepsilon, v^{-a}_\delta)$. For every matching μ that is stable under v^I_δ, there exists a matching μ' stable under $(v^a_\varepsilon, v^{-a}_\delta)$ such that:*

8 Valuations with size-dependent discounts also appear in auction and procurement settings (see Delacrétaz et al., 2019).

9 For a recent survey of the matching with constraints literature, see Kamada and Kojima (2017).

10 To embed hard constraints, such as a school's maximal physical capacity $N + \bar{n}$, it suffices to set the discount term $\delta_{N+\bar{n}+1}$ to a very high value (in this case, larger than the highest possible score).

11 The general family of valuation functions with *size-dependent discounts* of Delacrétaz et al. (2019) allows for various "*blocks*" of objects, where each *block* corresponds to, e.g., a different commodity. Ehlers et al. (2014) allowed for various student *types* that may be determined by, e.g., ethnicity or some other socioeconomic criteria. Here, for simplicity, we consider the *one-block* version of the size-dependent discounts valuation of Delacrétaz et al. (2019), corresponding to *same types* in the setting of Ehlers et al. (2014).

12 Our requirement that $\varepsilon^a_1 = \delta^a_1 = 0$ is natural because it means that both v^a_δ and v^a_ε correspond to the same valuations v^a over *individual* agents in $\bar{I}^a$.

1. μ' is preferred to μ by all agents on the other side of the market from a; and,
2. μ is preferred to μ' by all agents (other than a) on the same side of the market as a.

To prove Theorem 1, we appeal to a series of powerful comparative static results that Chambers and Yenmez (2017) proved for matching markets in which all agents have *path-independent* choice functions. We first show that if agents' valuations exhibit size-dependent discounts, then: *(i)* the induced choice functions are *path-independent*; and *(ii)* following a weak decrease in one of the discounts of an agent a, the resulting choice function is an *expansion* of a's initial choice function. Together with results of Chambers and Yenmez (2017), *(i)* and *(ii)* allow us to prove Theorem 1.

3.1. Size-dependent discounts induce path-independent choice functions

To show that size-dependent discounts induce path-independent choice functions, we prove that if the agents' valuations exhibit size-dependent discounts, then their choice functions satisfy the well-known *irrelevance of rejected agents* and *substitutability* conditions (Lemmata 1 and 2), which together imply *path-independence* (Lemma 3).

First, we note that valuations exhibiting size-dependent discounts induce choice functions that satisfy the *irrelevance of rejected agents* condition, which requires that an agent a's choice be unaffected by the removal of a set of agents that a does not choose.

Definition 1. An agent a's choice function C^a_δ satisfies the *irrelevance of rejected agents* condition if for all $A', A \subseteq \bar{I}^a$ such that $C^a_\delta(A) \subseteq A' \subseteq A$, we have $C^a_\delta(A) = C^a_\delta(A')$.[13]

Lemma 1. *If an agent's valuation exhibits size-dependent discounts, then his choice function satisfies the irrelevance of rejected agents condition.*

Proof. Suppose that we have $A', A \subseteq \bar{I}^a$ such that $C^a_\delta(A) \subseteq A' \subseteq A$, and suppose for the sake of seeking contradiction that $C^a_\delta(A') \neq C^a_\delta(A)$: then, since a's induced preferences are strict, we must have $v^a_\delta(C^a_\delta(A)) \neq v^a_\delta(C^a_\delta(A'))$. Then, the fact that $C^a_\delta(A) \subseteq A'$ implies that $v^a_\delta(C^a_\delta(A)) < v^a_\delta(C^a_\delta(A'))$ (because $C^a_\delta(A)$ can be chosen but is not optimal when the agents in A' are available). However, the fact that $C^a_\delta(A') \subseteq A' \subseteq A$ implies that $v^a_\delta(C^a_\delta(A)) > v^a_\delta(C^a_\delta(A'))$ (because $C^a_\delta(A')$ can be chosen but is not optimal when the agents in A are available), a contradiction. □

Next, we note that valuations exhibiting size-dependent discounts induce choice functions that are *substitutable* in the sense that whenever the set of agents available to choose from shrinks, all agents who were initially chosen and are still available remain chosen.

Definition 2. An agent a's choice function is *substitutable* if for any distinct $A', A \subseteq \bar{I}^a$, we have that $C^a_\delta(A' \cup A) = A'$ implies $a' \in C^a_\delta(\{a'\} \cup A)$ for any $a' \in A'$.[14]

Lemma 2 (*Delacrétaz et al., 2020*)**.** *If an agent's valuation exhibits size-dependent discounts, then his choice function is substitutable.*

Finally, a choice function is *path-independent* if, when the set of agents available to choose from is partitioned into two subsets, the choice from the initial set of agents coincides with the choice over the independent choices from each of the sets in the partition.

Definition 3. An agent a's choice function C^a_δ is *path-independent* if for every $A', A \subseteq \bar{I}^a$, we have $C^a_\delta(A' \cup A) = C^a_\delta(C^a_\delta(A') \cup C^a_\delta(A))$.[15]

By Lemma 1 of Chambers and Yenmez (2017), choice functions are path-independent if (and only if) they satisfy the irrelevance of rejected agents and substitutability conditions;[16] hence, the following result is immediate from our Lemmata 1 and 2.

Lemma 3. *If an agent's valuation exhibits size-dependent discounts, then his choice function is path-independent.*

3.2. Choice expansion under size-dependent discounts

Now, we prove Theorem 1 by way of comparative static results of Chambers and Yenmez (2017) for path-independent choice functions. We first show by induction that if an agent a's valuation exhibits size-dependent discounts, then a discount reduction from δ^a to ε^a induces a new choice function C^a_ε that is an *expansion* of C^a_δ, in the sense that for any set of agents on the other side of the market, each agent chosen under C^a_δ is also chosen under C^a_ε.

Definition 4. A choice function $\widehat{C}^a$ for agent a is an *expansion* of C^a_δ if for every $A \subseteq \bar{I}^a$, we have $\widehat{C}^a(A) \supseteq C^a_\delta(A)$.[17]

Lemma 4. *If an agent's valuation exhibits size-dependent discounts, then a discount reduction for that agent leads to a choice function that is an expansion of the agent's original choice function.*

Proof. First, we fix an agent a, a set of agents $A \subseteq \bar{I}^a$, and a discount profile δ. Without loss of generality, we index the agents in A in decreasing order of their individual values to a: $A = \{b_1, b_2, \ldots, b_{|A|}\}$ with $v^a(b_1) > v^a(b_2) > \cdots > v^a(b_{|A|})$.[18]

Let $\ell \equiv |C^a_\delta(A)|$ be the number of agents that a chooses from A. We show that $C^a_\delta(A) = \{b_1, \ldots, b_\ell\}$. Suppose to the contrary that $C^a_\delta(A) \neq \{b_1, \ldots, b_\ell\}$. Then, there exists an $n \leq \ell$ such that $b_n \notin C^a_\delta(A)$ and an $m > \ell$ such that $b_m \in C^a_\delta(A)$, so

$$v^a_\delta((C^a_\delta(A) \cup \{b_n\}) \setminus \{b_m\}) = v^a_\delta(C^a_\delta(A)) + v^a(b_n) - v^a(b_m).$$

As $n < m$, $v^a(b_n) > v^a(b_m)$. Therefore, we have

$$v^a_\delta((C^a_\delta(A) \cup \{b_n\}) \setminus \{b_m\}) > v^a_\delta(C^a_\delta(A)),$$

which contradicts the fact that $C^a_\delta(A)$ is the subset of A that maximizes v^a_δ.

Now, we let ε be a discount profile such that $\varepsilon \leq \delta$. To prove the lemma, we need to show that $C^a_\delta(A) \subseteq C^a_\varepsilon(A)$.

[13] Blair (1988), Alkan and Gale (2003), and Chambers and Yenmez (2017) use the term "consistency" instead of *irrelevance of rejected agents* (Aygün and Sönmez, 2013); we adopt the latter terminology to avoid any potential confusion with consistency as a normative requirement (see, e.g., Thomson, 2020).

[14] Substitutability is essential for establishing the existence of stable outcomes in various two-sided matching settings (see, e.g., Kelso and Crawford, 1982; Roth, 1984); moreover, under substitutability, as shown by Hatfield and Milgrom (2005), simple deferred-acceptance-like auctions can be used to find a solution. Beyond two-sided settings, *(full) substitutability* remains an essential condition (Hatfield et al., 2013, 2019, 2020); for a history of substitutability, see Hatfield et al. (2019).

[15] Path-independence, first formally introduced by Plott (1973), is a key property in the choice-theory literature; for a survey, see Moulin (1985).

[16] This result was initially stated in a different setting and without proof by Aizerman and Malishevski (1981, Corollary 2); that version of the result was then formally shown to be correct by Moulin (1985, Lemma 6).

[17] Chambers and Yenmez (2017) list a number of practical situations in which choice function expansions play an important role, such as in controlled school choice (Hafalir et al., 2013; Ehlers et al., 2014) and residency matching under constraints (Kamada and Kojima, 2015).

[18] As the induced preferences are strict, agent a does not have the same value for any two agents in A.

We have shown that $C^a_\delta(A) = \{b_1, \ldots, b_\ell\}$. Letting $\ell' \equiv |C^a_\varepsilon(A)|$, analogous reasoning shows that $C^a_\varepsilon(A) = \{b_1, \ldots, b_{\ell'}\}$; therefore, we need to show that $\ell' \geq \ell$. Suppose to the contrary that

$$\ell' < \ell. \tag{2}$$

Thus, we must have

$$v^a(b_{\ell'+1}) < \varepsilon^a_{\ell'+1}, \tag{3}$$

as otherwise we would have

$$v^a_\varepsilon(C^a_\varepsilon(A) \cup \{b_{\ell'+1}\}) = v^a_\varepsilon(C^a_\varepsilon(A)) + v^a(b_{\ell'+1}) - \varepsilon^a_{\ell'+1} > v^a_\varepsilon(C^a_\varepsilon(A)),$$

contradicting the fact that $C^a_\varepsilon(A)$ is the subset of A that maximizes v^a_ε.

Our hypothesis (2) implies that

$$\ell' + 1 \leq \ell; \tag{4}$$

hence, as $\varepsilon \leq \delta$ and $\delta^a_1 \leq \delta^a_2 \leq \cdots \leq \delta^a_{|\bar{I}^a|}$, (3) implies that

$$\delta^a_\ell \overset{b_\ell \in C^a_\delta(A)}{\leq} v^a(b_\ell) \overset{(4)}{\leq} v^a(b_{\ell'+1}) \overset{(3)}{<} \varepsilon^a_{\ell'+1} \overset{(\varepsilon \leq \delta)}{\leq} \delta^a_{\ell'+1} \overset{(4)}{\leq} \delta^a_\ell,$$

a contradiction. □

Given Lemmata 3 and 4, Theorem 1 follows immediately from Theorem 2 of Chambers and Yenmez (2017), which shows that an expansion induces the desired comparative static so long as all agents' choice functions are path-independent. Alternatively, given Lemmata 1, 2, and 4, Theorem 1 also follows immediately from Theorem 7 of Pycia and Yenmez (2019).[19]

4. Discussion

We can apply our Theorem 1 iteratively to show that following a discount reduction for every agent on one side of the market, all agents on the other side become better off—a result that mirrors Corollary 2 of Chambers and Yenmez (2017).

Corollary 1. *Suppose that all agents' valuations exhibit size-dependent discounts and consider any profile of discounts δ with associated valuation profile v^I_δ. For each firm $f \in F$, consider a discount reduction from δ^f to ε^f and denote the post-discount-reduction valuation profile by $(v^F_\varepsilon, v^W_\delta)$. Then, for any matching μ that is stable under v^I_δ, there exists a matching μ' that is stable under $(v^F_\varepsilon, v^W_\delta)$ such that every worker w prefers μ' to μ.*

By combining our results in Theorem 1 with Corollary 1 of Chambers and Yenmez (2017), we obtain a sharpening of our main result that speaks to deferred-acceptance-like mechanisms by comparing side-optimal stable matchings.[20] Formally, we say that a stable matching μ is *firm-optimal* if each firm prefers it to any other stable matching μ'. Likewise, a stable matching μ is *worker-optimal* if every worker prefers μ to any other stable matching μ'.

Corollary 2. *Suppose that all agents' valuations exhibit size-dependent discounts and consider any profile of discounts δ with associated valuation profile v^I_δ. Fix any firm $f \in F$, and consider any discount reduction from δ^f to ε^f, with associated post-discount reduction valuation profile $(v^f_\varepsilon, v^{-f}_\delta)$. Let μ^F and μ^W, respectively, be the firm- and worker-optimal stable matchings under v^I_δ. Analogously, let $\bar{\mu}^F$ and $\bar{\mu}^W$ be the firm- and worker-optimal stable matchings under $(v^f_\varepsilon, v^{-f}_\delta)$.[21] Then, the following hold:*

1. *every worker prefers $\bar{\mu}^W$ to μ^W, and every firm other than f prefers μ^W to $\bar{\mu}^W$; and,*
2. *every worker prefers $\bar{\mu}^F$ to μ^F, and every firm other than f prefers μ^F to $\bar{\mu}^F$.*

It is clear from our argument for Theorem 1 that the result extends to settings in which agents other than the one undergoing a discount reduction have general path-independent choice functions—it is not necessary that all agents' preferences exhibit size-dependent discounts.

Moreover, Theorem 1 is "sharp" in the sense that a simultaneous discount reduction for agents on different sides of the market may have an ambiguous effect. Indeed, suppose that *all* agents' valuations exhibit size-dependent discounts and consider any profile of discounts δ with associated valuation profile v^I_δ. We consider simultaneous discount reductions on both sides of the market fixing, say, $f \in F$ and $w \in W$, and taking discount reductions from δ^f to ε^f and from δ^w to ε^w. As the following example illustrates, with discount reductions on both sides, the conclusion of Theorem 1 may not hold.

Example 1. Let $F = \{f_1, f_2, f_3\}$, $W = \{w_1, w_2, w_3\}$, and $I = F \cup W$. The valuations over individual agents are as follows:

$v^{f_1}(w_1) = 1$	$v^{f_2}(w_1) = -1$	$v^{f_3}(w_1) = 1$	$v^{w_1}(f_1) = 1$	$v^{w_2}(f_1) = 1$	$v^{w_3}(f_1) = 3$
$v^{f_1}(w_2) = 3$	$v^{f_2}(w_2) = 3$	$v^{f_3}(w_2) = 3$	$v^{w_1}(f_2) = -1$	$v^{w_2}(f_2) = 3$	$v^{w_3}(f_2) = 1$
$v^{f_1}(w_3) = 5$	$v^{f_2}(w_3) = 5$	$v^{f_3}(w_3) = 5$	$v^{w_1}(f_3) = 5$	$v^{w_2}(f_3) = 5$	$v^{w_3}(f_3) = 5$

For each agent $a \in I$, let the vector of discounts be $\delta^a = (0, 9, 10)$ and the associated valuation over sets of agents be v^a_δ. Recall that valuations over sets of agents are determined by using the discount vector to extend the valuations over individual agents (see (1)); for example, the value of agent w_1 for being matched to both f_1 and f_3 is

$$v^{w_1}_\delta(\{f_1, f_3\}) = v^{w_1}(f_1) + v^{w_1}(f_3) - \delta^{w_1}_1 - \delta^{w_1}_2 = 1 + 5 - 0 - 9 = -3.$$

For the profile of discounts $\delta = (\delta^a)_{a \in I}$ with associated valuation profile $v^I_\delta = (v^a_\delta)_{a \in I}$, there is a unique stable matching μ,[22] under which

$$\mu(f_1) = \{w_1\}, \mu(f_2) = \{w_2\}, \mu(f_3) = \{w_3\}.$$

We now consider the effects of a simultaneous (and identical) discount reduction for f_3 and w_3; the discounts for all other agents are left unchanged. Let $\varepsilon^{f_3} = \varepsilon^{w_3} = (0, 2, 10)$, and for each $a \in I \setminus \{f_3, w_3\}$, let $\varepsilon^a = \delta^a$. For the profile of discounts $\varepsilon = (\varepsilon^a)_{a \in I}$ with associated valuation profile $v^I_\varepsilon = (v^a_\varepsilon)_{a \in I}$, there is a unique stable matching μ', i.e.,

$$\mu'(f_1) = \{w_3\}, \mu'(f_3) = \{w_2, w_3\}, \mu'(w_3) = \{f_1, f_3\},$$

under which agents f_2 and w_1 are unmatched.

Thus, matching μ' is preferred to μ by agents f_1, f_3, w_2, w_3, while matching μ is preferred to μ' by agents f_2 and w_1.

[19] To see this, note that Pycia and Yenmez (2019) consider a more general matching model with externalities, and thus have to formulate both their irrelevance of rejected contracts and substitutability conditions with respect to a reference set; in the absence of externalities, their model and conditions coincide with ours, and their comparative static result is equivalent to that of Chambers and Yenmez (2017).

[20] For a description of a many-to-many deferred acceptance mechanism, see Chambers and Yenmez (2017).

[21] These extremal stable matchings exist because size-dependent discounts induce substitutable choice functions (Lemma 2).

[22] To see that μ is the unique stable matching, note that: (i) the discounts are so large that each agent chooses at most one partner; (ii) the valuations over individual agents induce positive assortative matching in which the worker and firm with the first-, second-, and third-highest values for each other are matched.

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