CHAPTER FOURTEEN

Generalized Matching: Contracts and Networks

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14.1 Introduction

Many real-world settings incorporate features that go beyond the standard matching setting described in Chapter 1. In particular, whether a given partner is acceptable may depend on the terms of the relationship, such as wages, hours, specific job responsibilities, and the like. In [4] and [33] it was shown that it is possible to extend classical matching frameworks to determine not only who matches with whom but also additional terms – such as wages – that specify the full "contractual" terms of exchange.

Moreover, in many settings, one side of the market has a multi-unit demand which cannot always be represented by a simple ranking of contracts and a capacity constraint. For instance, a hospital may have both a research position and a clinical position (as specified in the contract), and rank candidates differently for each position. A firm may have two positions and one ranking of candidates but may also desire that, if possible, at least one hire have a particular feature, such as fluency in a foreign language. And a baseball team may prefer different catchers depending on the pitchers it has available.

In [10], [19], [24], [33], and [40] it was shown that, when buyers have multi-unit demand, some form of *substitutability* in buyers' preferences is key to ensuring the existence of stable outcomes. Substitutability requires that no two contracts are complements, in the sense that if a contract is rejected given some opportunity set, that contract will still be rejected as more opportunities become available. Thus, both our hospital and our firm above have substitutable preferences, while our baseball team does not. That said, certain types of complementarities can be accommodated under many-to-one matching with contracts. The many-to-one matching with contracts framework unified matching and auctions, and has led to a number of high-profile real-world applications such as the reorganization of the US Army's cadet-branch matching system [14], [41], [42], and the Israeli Psychology Master's Match [18].

The theory of matching with contracts extends to cover two-sided settings in which agents on both sides may have multi-unit demand [9], [10], [22], [34]. For instance, buyers in an auction may demand multiple goods when the auctioneer has many items for sale. Most results from many-to-one matching with contracts extend naturally to this setting.

The theory can also be extended to more complex market environments. In [39] supply-chain settings were considered in which intermediaries may both buy from upstream firms and sell to downstream firms, so the firm is no longer exclusively just a buyer or just a seller; see also [21] and [45]. The appropriate definition of substitutability for such settings is somewhat subtle: it requires that an intermediary treats contracts in which he is a buyer as substitutes, and treats contracts in which he is a seller as substitutes, but treats contracts in which he is a buyer as complementary with contracts in which he is a seller.[1] Indeed, the theory extends beyond supply chains to arbitrarily complex trading networks, in which agents may buy from and sell to any other agent. However, the existence of stable outcomes is not immediate in such settings. In [25] it was shown that stable outcomes do exist when transfers are encoded into the contracts, agents' preferences are quasilinear in the transfers, and agents' preferences are substitutable. Meanwhile it was shown in [11], [12], [13] that stable outcomes exist even without transferable utility, as long as payments are not affected by distortionary frictions.

Section 14.2 introduces a general matching with contracts framework. Section 14.3 then considers two-sided matching settings and gives a characterization of when stable outcomes can be found in such settings. Section 14.4 examines the supply chain and trading network settings, generalizing many results of Section 14.3. Finally, Section 14.5 extends the matching with contracts framework to allow for transfers.

14.2 The Framework

Consider a finite set of *agents* I, a finite set of contractual *terms* T, and a set of bilateral *contracts* $X \subseteq I \times I \times T$. A contract $x = (s, b, t) \in X$ represents a relationship between two agents, a "seller" $s \in I$ and a "buyer" $b \in I \setminus \{s\}$, under terms $t \in T$. For example, in an exchange economy with indivisible goods and no monetary transfers, a contract $x = (s, b, t)$ would represent the transfer of some (unit of) good t from seller s to buyer b; alternatively, the terms t could represent any combination of goods transferred, services provided, and a price or wage. For any given contract $x = (s, b, t) \in X$, we denote by $\mathsf{s}(x)$ the associated seller s, by $\mathsf{b}(x)$ the associated buyer b, and by $\mathsf{t}(x)$ the associated contract terms t. Given any set of contracts $Y \subseteq X$, we denote by $Y_{i\to} \equiv \{y \in Y : \mathsf{s}(y) = i\}$ the set of contracts for which i is a seller, by $Y_{\to i} \equiv \{y \in Y : \mathsf{b}(y) = i\}$ the set of contracts in which i is a buyer, and by $Y_i \equiv Y_{i\to} \cup Y_{\to i}$ the set of all contracts for which agent i is involved.

Each agent $i \in I$ is endowed with a *choice correspondence* C^i that specifies which sets of contracts agent i would choose to sign from any fixed set of available contracts, and so $C^i(Y) \subseteq \wp(Y_i)$ for all $Y \subseteq X$, where $\wp$ denotes the power set of Y_i. We say that agent i has *unit demand* if for all $Y \subseteq X$ we have that $|Z| \leq 1$ for all $Z \in C^i(Y)$. Whenever the choice correspondence is single-valued on all inputs, i.e., $C^i(Y) = \{Z\}$ for all $Y \subseteq X$, we call C^i a *choice function* and write $C^i(Y) = Z$.

In our examples it will often be helpful to describe choice correspondences as arising from preference rankings over sets of contracts. A weak preference relation $\succeq_i$ for agent i over subsets of X_i *induces* a choice correspondence C^i for i, under which

[1] If each contract specifies the transfer of an underlying object, then this definition of substitutability is natural; it requires that an agent treat the underlying objects as substitutes [27].

$$C^i(Y) = \max_{\succeq_i}\{Z \subseteq X_i : Z \subseteq Y\},$$

where by $\max_{\succeq_i}$ we mean the maxima with respect to the ordering $\succeq_i$; that is, $C^i(Y)$ contains all subsets of Y that are most preferred with respect to $\succeq_i$. When an agent has a single-valued unit-demand choice correspondence (i.e., a unit-demand choice function), this can be induced by a preference relation over contracts involving that agent.

14.3 Two-Sided Matching with Contracts

In this section we focus on two-sided matching markets, that is, we assume that the set of agents I can be partitioned into a set of *buyers* B and *sellers* S such that, for every contract $x \in X$, we have $\mathsf{s}(x) \in S$ and $\mathsf{b}(x) \in B$. We also assume throughout this section that choice correspondences are single-valued on all inputs. We may thus define the aggregate choice functions of buyers and sellers: for each $Y \subseteq X$, let $C^S(Y) = \cup_{s \in S} C^s(Y)$ and $C^B(Y) = \cup_{b \in B} C^b(Y)$, respectively. Finally, we also assume throughout this section that every choice function satisfies the *irrelevance of rejected contracts condition*, i.e., that i considers rejected contracts to be irrelevant in the sense that, for all $Y \subseteq X$ and all $x \in Y \setminus C^i(Y)$, we have that $C^i(Y \setminus \{x\}) = C^i(Y)$.[2]

14.3.1 Many-to-Many Matching with Contracts

One important condition on preferences that guarantees the existence of stable outcomes is *(gross) substitutability*.

Definition 14.1. The choice function of agent $i \in I$ is *(gross) substitutable* if for all $Y \subseteq X$ and all contracts $x, z \in X$, we have that $z \in C^i(Y \cup \{x, z\})$ implies $z \in C^i(Y \cup \{z\})$.

Substitutability may equivalently be defined by considering how the set of contracts that an agent rejects (i.e., does not choose) varies across different inputs. Formally, for an agent $i \in I$ and a set of contracts $Y \subseteq X$, let $R^i(Y) \equiv Y \setminus C^i(Y)$ be the *rejected set*. The condition in Definition 14.1 is then equivalent to requiring that, for all $Y, Z \subseteq X$ such that $Y \subseteq Z$, we have $R^i(Y) \subseteq R^i(Z)$. Note that if the choice function of every buyer (seller) is substitutable then the aggregate choice function C^B (C^S) is substitutable.

Next, we introduce the standard notion of *pairwise stability*, which parallels the definition of stability from Chapter 1.

Definition 14.2. An outcome $A \subseteq X$ is *pairwise stable* if it is:

1. *Individually rational*: For all $i \in I$, $C^i(A) = A_i$.
2. *Pairwise unblocked*: There does not exist a buyer–seller pair $(b, s) \in B \times S$ and a contract $z \in (X_b \cap X_s) \setminus A$ such that $z \in C^b(A \cup \{z\}) \cap C^s(A \cup \{z\})$.

[2] See [2].

The first main result of this section shows that pairwise stable outcomes are guaranteed to exist when all agents' choice functions are (gross) substitutable. Moreover, the set of pairwise stable outcomes is a lattice with respect to the order $\sqsupseteq_B$, where $Y \sqsupseteq_B Z$ if $Y_b = C^b(Y \cup Z)$ for all $b \in B$, and with respect to the order $\sqsupseteq_S$, where $Z \sqsupseteq_S Y$ if $Z_s = C^s(Y \cup Z)$ for all $s \in S$. And, as in Chapter 1, among pairwise stable outcomes these two orders are "opposed," in the sense that if both Y and Z are stable and $Y \sqsupseteq_B Z$ then $Z \sqsupseteq_S Y$.

Theorem 14.3 [22]. *Assume that the choice functions of all agents are substitutable. Then the set of pairwise stable outcomes is a non-empty lattice with respect to $\sqsupseteq_B$ and $\sqsupseteq_S$; in particular, there exists a buyer-optimal/seller-pessimal (as well as a seller-optimal/buyer-pessimal) pairwise stable outcome.*

Theorem 14.3 generalizes the existence results of Chapters 1 and 3. Moreover, incorporating contractual terms substantially extends the domain of applications we can consider, including such settings as matching with wages [33] and multi-unit demand auctions [36].

Proof of Theorem 14.3 To prove Theorem 14.3, we introduce a *generalized deferred acceptance* (DA) operator, show that the set of pairwise stable outcomes corresponds to the set of fixed points of this operator, and then use Tarski's fixed-point theorem to establish that the set of fixed points of the generalized DA operator is a non-empty lattice.

Given two sets of contracts $X^S, X^B \subseteq X$, we define the generalized DA operator as follows:

$$\begin{aligned} \Phi(X^S, X^B) &= (\Phi^S(X^B), \Phi^B(X^S)), \\ \Phi^S(X^B) &= \{x \in X : x \in C^B(X^B \cup \{x\})\}, \\ \Phi^B(X^S) &= \{x \in X : x \in C^S(X^S \cup \{x\})\}. \end{aligned} \tag{14.1}$$

Here, we can think of X^S (at each iteration of the operator) as the set of contracts available to sellers (and X^B as the set of contracts available to buyers). To determine whether a contract x is available to sellers (in the next iteration) given X^B, we ask whether x would be chosen by buyers if $X^B \cup \{x\}$ were available to buyers; analogously, to determine whether a contract x is available to buyers (in the next iteration) given X^S, we ask whether x would be chosen by sellers if $X^S \cup \{x\}$ were available to sellers.

Lemma 14.4. *If (X^S, X^B) is a fixed point of Φ then $A = X^S \cap X^B$ is pairwise stable, $C^B(X^B) = A$, and $C^S(X^S) = A$. Furthermore, if all agents have substitutable preferences and A is pairwise stable then $\Phi(A, A)$ is a fixed point of Φ.*

Proof We start by assuming that (X^S, X^B) is a fixed point of Φ and, letting $A = X^S \cap X^B$, we show that $C^B(X^B) = A$, that $C^S(X^S) = A$, and that A is pairwise stable.

- $C^B(X^B) = A$: *Let $x \in A$ be arbitrary. Since $x \in X^S$ and $X^S = \Phi^S(X^B)$, we have $x \in C^B(X^B \cup \{x\})$. Since $x \in X^B$, we obtain $x \in C^B(X^B)$. Thus, since x was arbitrary, we have $A \subseteq C^B(X^B)$.*

Next, consider some $x \in X^B \setminus X^S$. If $x \in C^B(X^B)$, then we have $x \in C^B(X^B \cup \{x\})$ and therefore $x \in \Phi^S(X^B)$. Since $x \notin X^S$, we obtain a contradiction to our assumption that $X^S = \Phi^S(X^B)$.

- $C^S(X^S) = A$: *An argument analogous to the preceding argument shows that* $C^S(X^S) = A$.
- *A is pairwise stable: Individual rationality of A for buyers follows since* $C^B(X^B) = A$ *(from our preceding argument) and thus – since agents' choice functions satisfy the irrelevance of rejected contracts condition – we have that* $C^B(A) = A$. *We now show that A is pairwise unblocked. Let* $z \in X \setminus A$ *be arbitrary and assume that* $z \notin X^S$*; the case where* $z \notin X^B$ *is analogous. We will argue that* $z \notin C^B(A \cup \{z\})$*; if so, then* $\{z\}$ *does not block A. Since* $X^S = \Phi^S(X^B)$*, we have that* $z \notin C^B(X^B \cup \{z\})$*; thus, since* $C^B(X^B) = A$*, by the irrelevance of rejected contracts condition,* $z \notin C^B(A \cup \{z\})$*. Next, we assume that A is pairwise stable, we let* $(X^S, X^B) = \Phi(A, A)$*, and we show that* $A = X^S \cap X^B$ *and* (X^S, X^B) *is a fixed point of* Φ.
- $A = X^S \cap X^B$: *If* $A \nsubseteq X^S \cap X^B$*, the definition of* Φ *immediately implies that A is not individually rational. If there were a contract* $z \in (X^S \cap X^B) \setminus A$*, we would have* $z \in C^B(A \cup \{z\})$ *(as* $z \in X^S$*) and* $z \in C^S(A \cup \{z\})$ *(as* $z \in X^B$*) so that A would be blocked by* $\mathsf{b}(z)$ *and* $\mathsf{s}(z)$.
- (X^S, X^B) *is a fixed point of* Φ: *We show that* $\Phi^S(X^B) = X^S$*; the fact that* $\Phi^B(X^S) = X^B$ *follows by an analogous argument. We show first that* $\Phi^S(X^B) \subseteq X^S$*. Let* $y \in \Phi^S(X^B)$ *be arbitrary. By the definition of* Φ*, we have that* $y \in C^B(X^B \cup \{y\})$*. By substitutability, we obtain that* $y \in C^B(A \cup \{y\})$ *and therefore* $y \in \Phi^S(A) = X^S$*. Now, we argue that* $X^S \subseteq \Phi^S(X^B)$*. Let* $y \in X^S$ *be arbitrary. Since* $X^S = \Phi^S(A)$*, we obtain that* $y \in C^B(A \cup \{y\})$*. If* $y \notin \Phi^S(X^B)$*, we would have* $y \notin C^B(X^B \cup \{y\})$ *and, by the irrelevance of rejected contracts condition, there is some* $z \in X^B \setminus A$ *such that* $z \in C^B(X^B \cup \{y\})$*. By substitutability, we have that* $z \in C^B(A \cup \{z\})$ *and thus* $z \in X^S$*. Hence,* $z \in X^S \cap X^B$ *and this contradicts* $X^S \cap X^B = A$. □

For the remainder of the proof of Theorem 14.3, we introduce the following order on $X \times X$: $(X^S, X^B) \vdash (\tilde{X}^S, \tilde{X}^B)$ if and only if $X^S \subseteq \tilde{X}^S$ and $X^B \supseteq \tilde{X}^B$.

We show first that Φ is isotone with respect to $\vdash$. For that purpose, take any pair $(X^S, X^B), (\tilde{X}^S, \tilde{X}^B) \in X \times X$ such that $(X^S, X^B) \vdash (\tilde{X}^S, \tilde{X}^B)$. We need to show that $\Phi(X^S, X^B) \vdash \Phi(\tilde{X}^S, \tilde{X}^B)$, or $\Phi^S(X^B) \subseteq \Phi^S(\tilde{X}^B)$ and $\Phi^B(X^S) \supseteq \Phi^B(\tilde{X}^S)$. To show that $\Phi^S(X^B) \subseteq \Phi^S(\tilde{X}^B)$, take some $y \in X$ such that $y \in C^B(X^B \cup \{y\})$. By substitutability and the fact that $X^B \supseteq \tilde{X}^B$, we immediately obtain $y \in C^B(\tilde{X}^B \cup \{y\})$ and thus $y \in \Phi^S(\tilde{X}^B)$. The argument to show that $\Phi^B(X^S) \supseteq \Phi^B(\tilde{X}^S)$ is completely symmetric.

Since Φ is isotone with respect to $\vdash$, Tarski's fixed-point theorem implies that the set of fixed points is a non-empty lattice with respect to $\vdash$.

To complete the proof, we now show that the set of pairwise stable outcomes is a lattice with respect to the order $\sqsupseteq_S$. Take any two fixed points $(X^S, X^B), (\tilde{X}^S, \tilde{X}^B) \in X \times X$ of Φ such that $(X^S, X^B) \vdash (\tilde{X}^S, \tilde{X}^B)$. Let $A = X^S \cap X^B$ and $\tilde{A} = \tilde{X}^B \cap \tilde{X}^S$. We claim that $\tilde{A} \sqsupseteq_S A$. By the first part of Lemma 14.4, we have that $C^B(X^B) = A$ and $C^B(\tilde{X}^B) = \tilde{A}$. Since $X^B \supseteq \tilde{X}^B$, we obtain that $C^B(A \cup \tilde{A}) = A$. A similar argument establishes that $C^S(A \cup \tilde{A}) = \tilde{A}$.

Hence, the lattice property of the set of stable outcomes follows from the lattice property of the set of fixed points of Φ. □

Next, we define a concept of stability that allows arbitrary groups of agents to coordinate in order to block some outcome.

Definition 14.5. An outcome $A \subseteq X$ is *stable* if it is:

1. *Individually rational*: For all $i \in I$, $C^i(A) = A_i$.
2. *Unblocked*: There does not exist a non-empty set of contracts $Z \subseteq X \setminus A$ such that $Z \subseteq C^B(A \cup Z) \cap C^S(A \cup Z)$.

Note that while pairwise stability requires only the absence of blocks consisting of a single contract, stability requires the absence of blocking sets of contracts; hence, by definition, stability is more stringent than pairwise stability.

Our next result shows that under substitutability, pairwise stability and stability are equivalent.

Theorem 14.6 [22]. *If all choice functions are substitutable, then any pairwise stable outcome is stable.*

It turns out that substitutability is necessary in a maximal domain sense for the guaranteed existence of stable outcomes.[3] Consider markets in which the contract set is *exhaustive* in the sense that, for each pair $(b, s) \in B \times S$, there exists a contract $x \in X$ such that $\mathsf{b}(x) = b$ and $\mathsf{s}(x) = s$.

Theorem 14.7 [22]. *Suppose that there are at least two sellers and the contract set is exhaustive. If the choice function of agent $s \in S$ is not substitutable then there exist substitutable choice functions for the other agents such that no stable outcome exists.*

The following example shows how to construct an economy without a stable outcome given an agent with non-substitutable preferences; the proof of Theorem 14.7 generalizes the structure of this example.

Example 14.8 [22]. Consider a seller s with the choice function C^s induced by the preference ordering

$$\succ_s : \{x, y\} \succ \varnothing,$$

where $\mathsf{b}(x) \neq \mathsf{b}(y)$. Note that C^s is not substitutable, as $C^s(\{x, y\}) = \{x, y\}$ while $C^s(\{y\}) = \varnothing$. Now, suppose that there exists another seller s' and two contracts x' and y' with s' such that $\mathsf{b}(x') = \mathsf{b}(x)$ and $\mathsf{b}(y') = \mathsf{b}(y)$, and suppose that the choice function of s' is induced by the preference relation

$$\succ_{s'} : \{y'\} \succ \{x'\} \succ \varnothing.$$

[3] The existence of stable outcomes in the presence of non-substitutability can sometimes be obtained in large market frameworks like those of Chapter 16.

Finally, suppose that $\mathsf{b}(x)$ has choice function $C^{\mathsf{b}(x)}$ induced by the preference ordering

$$\succ_{\mathsf{b}(x)}: \{x'\} \succ \{x\} \succ \varnothing$$

and $\mathsf{b}(y)$ has choice function $C^{\mathsf{b}(y)}$ induced by the preference ordering

$$\succ_{\mathsf{b}(y)}: \{y\} \succ \{y'\} \succ \varnothing.$$

The outcome $\{x, y\}$ is not stable, as $\{x'\}$ is a blocking set. However, for any other individually rational allocation A, we must have that $A_s = \varnothing$; but then, either $\{x, y\}$ is a blocking set (if $x' \notin A$) or $\{y'\}$ is a blocking set (if $x' \in A$).

We now introduce a second restriction on agents' choice functions which requires that the *number* of chosen contracts weakly increases when the set of available contracts increases (in a superset sense).

Definition 14.9. The choice function of a buyer $i \in B$ (seller $i \in S$) satisfies the *law of aggregate demand (supply)* if for any pair of contracts $Y, Z \subseteq X$ such that $Y \subseteq Z$, we have that $|C^i(Y)| \leq |C^i(Z)|$.

When combined with substitutability, the laws of aggregate supply and demand allow us to generalize the *rural hospitals theorem* of Chapter 1 to our many-to-many setting.

Theorem 14.10 [22]. *If the choice functions of all agents are substitutable and satisfy the laws of aggregate supply and demand, then for each agent $i \in I$, the number of contracts that i signs is invariant across all stable outcomes.*

One can use the preceding theorem to show that a mechanism that picks the buyer-optimal or seller-optimal stable outcome is dominant-strategy incentive compatible, or *strategy-proof*, for all unit-demand buyers or sellers, respectively.

Theorem 14.11 [22]. *If the choice functions of all agents are substitutable and satisfy the laws of aggregate supply and demand then the buyer-optimal stable mechanism is strategy-proof for all unit-demand buyers and the seller-optimal stable mechanism is strategy-proof for all unit-demand sellers.*

Theorem 14.11 generalizes the strategy-proofness result of Chapter 1 to allow for substitutable preferences for the other side of the market. However, the next example shows that the incentive-compatibility results do not extend beyond the unit-demand case.

Example 14.12 [24]. Consider a seller s with the choice function C^s induced by the preference ordering

$$\succ_s: \{x, y\} \succ \{x, z\} \succ \{y, z\} \succ \{x\} \succ \{y\} \succ \{z\} \succ \varnothing;$$

that is, s prefers $\mathsf{b}(x)$ to $\mathsf{b}(y)$ to $\mathsf{b}(z)$ and desires at most two contracts. Additionally, there is a seller s' with the unit-supply choice function $C^{s'}$ induced by the preference ordering

$$\succ_{s'}: \{y'\} \succ \{x'\} \succ \{z'\} \succ \varnothing,$$

where $\mathsf{b}(x) = \mathsf{b}(x') \neq \mathsf{b}(y) = \mathsf{b}(y') \neq \mathsf{b}(z) = \mathsf{b}(z') \neq \mathsf{b}(x)$.

The choice functions of the buyers are induced by the preferences

$$\succ_{\mathsf{b}(x)}: \{x'\} \succ \{x\} \succ \varnothing$$
$$\succ_{\mathsf{b}(y)}: \{y\} \succ \{y'\} \succ \varnothing$$
$$\succ_{\mathsf{b}(z)}: \{z\} \succ \{z'\} \succ \varnothing.$$

The only stable outcome is $\{x', y, z\}$. However, if s were to report

$$\hat{\succ}_s: \{x, z\} \succ \{x\} \succ \{z\} \succ \varnothing,$$

then the only stable outcome under the reported preferences would be $\{x, y', z\}$; this outcome is preferred by s even under the preferences $\succ_s$.

14.3.2 Many-to-One Matching with Contracts

A case of special interest is the many-to-one matching with contracts setting, introduced and developed by [10], [24], and [33]; see also [40]. In this setting, buyers have unit demand. It is immediate that stable outcomes exist in this setting when sellers' choice functions are substitutable (Theorem 14.3) and, when sellers' choice functions satisfy the law of aggregate supply, the buyer-optimal stable mechanism is strategy-proof for buyers (Theorem 14.11).

However, it is no longer the case that substitutability is necessary (even in the maximal domain sense of Theorem 14.7) for the existence of stable outcomes.

Example 14.13 [19]. Consider a seller s with the choice function C^s induced by the preference ordering

$$\succ_s: \{x, y\} \succ \{\tilde{x}\} \succ \{x\} \succ \{y\} \succ \varnothing,$$

where $\mathsf{b}(x) = \mathsf{b}(\tilde{x}) \neq \mathsf{b}(y)$. Note that C^s is not substitutable, as $C^s(\{\tilde{x}, x\}) = \{\tilde{x}\}$ while $C^s(\{\tilde{x}, x, y\}) = \{x, y\}$.

However, a stable outcome always exists as long as other sellers have substitutable choice functions. To see this, note that $\mathsf{b}(x)$ must either prefer $\{x\}$ to $\{x'\}$ or prefer $\{x'\}$ to $\{x\}$. In the former case we can treat the choice function of s *as if* it were induced by

$$\succ_s: \{x, y\} \succ \{x\} \succ \{y\} \succ \varnothing,$$

and these preferences induce a substitutable choice function; moreover, the outcome of a *buyer-proposing deferred acceptance mechanism* (under the new preferences) will be stable (with respect to the original preferences), as x will never be rejected, and so $\mathsf{b}(x)$ must obtain a contract at least as good as x. In the latter case, we can treat the choice function of s *as if* it were induced by[4]

$$\succ_s: \{\tilde{x}\} \succ \{y\} \succ \varnothing,$$

[4] Here, by the buyer-proposing mechanism, we mean the fixed point of the generalized DA operator (14.1) obtained by starting at $(\varnothing, X)$ and iterating.

and these preferences also induce a substitutable choice function. Moreover, the outcome of a buyer-proposing deferred acceptance mechanism (under the new preferences) will be stable (with respect to the original preferences), as $\tilde{x}$ will never be rejected, and so $\mathsf{b}(x)$ must obtain a contract at least as good as $\tilde{x}$.

14.3.2.1 Weakened Substitutability Conditions

Examples like that in Example 14.13 have motivated the search for conditions on seller preferences that ensure the existence of stable outcomes. A number of weakened substitutability conditions that guarantee the existence of stable outcomes have been found. Moreover, many of these conditions guarantee the existence of a stable and strategy-proof (for unit demand buyers) mechanism. In [20], for example, the authors identified the following condition, called *unilateral substitutability*.

Definition 14.14. The choice function of a seller s is *unilaterally substitutable* if, for all x, z such that $\mathsf{b}(x) \neq \mathsf{b}(z)$ and all $Y \subseteq X \setminus (X_{\mathsf{b}(x)} \cup X_{\mathsf{b}(z)})$, we have that

$$z \notin C^s(\{x\} \cup Y \cup \{z\}) \setminus C^b(Y \cup \{z\}).$$

In [20] it was shown that when the choice function of each seller is unilaterally substitutable, a stable outcome always exists; moreover, a stable outcome can be found by any *cumulative offer mechanism* (and, in fact, the outcome of the cumulative offer mechanism does not depend on the ordering used by the mechanism).[5] Moreover, when the choice function of each seller also satisfies the law of aggregate supply, any cumulative offer mechanism is strategy-proof.

Later, a more general condition under which stable outcomes are guaranteed to exist – *substitutable completability* – was introduced by [23]. Substitutable completion interprets certain non-substitutable choice functions in many-to-one matching with contracts as substitutable choice functions in the setting of many-to-many matching with contracts. In the setting of Example 14.13, for instance, we can think of "completing" the choice function of buyer b as allowing b to choose the

[5] For any ordering $\vdash$ over the set of contracts and preferences $\succ$ for the (unit-demand) buyers, the outcome of the *cumulative offer mechanism* is determined by the following algorithm:

Step 0: The set of contracts *available* to the sellers is $A^0 \equiv \varnothing$.
Step $t \geq 1$: Construct the set

$$U^t \equiv \{x \in X \setminus A^{t-1} : \mathsf{b}(x) \notin \mathsf{b}(C^S(A^{t-1})),\ C^{\mathsf{b}(x)}(\{x\}) = x,$$
$$\text{and } \nexists z \in (X_{\mathsf{b}(x)} \setminus A^{t-1}) \text{ such that } C^{\mathsf{b}(x)}(\{x, z\}) = z\}.$$

If U^t is empty then the algorithm terminates and the outcome is $C^S(A^{t-1})$; otherwise, $A^t \equiv A^{t-1} \cup \{y\}$, where y is the highest-ranked contract in U^t according to $\vdash$.

Here, the set U^t is composed of contracts x such that:

1. the contract x has not yet been offered;
2. the buyer of x does not have any contract chosen by the sellers from A^{t-1};
3. the buyer of x finds x acceptable; and
4. the buyer of x does not have any other not-yet-offered contract z that they prefer to x.

(infeasible) set of contracts $\{x, \tilde{x}\}$ whenever it is available, i.e., to have a choice function $\hat{C}^s$ induced by

$$\hat{\succ}_s\colon \{x,\tilde{x}\} \succ \{x,y\} \succ \{\tilde{x}\} \succ \{x\} \succ \{y\} \succ \varnothing. \tag{14.2}$$

Note that $\hat{C}^s$ is substitutable; thus, by Theorem 14.3, there must exist a stable outcome A with respect to the completed choice function. However, since $\mathsf{b}(x)$ has unit demand, A cannot involve both x and $\tilde{x}$; hence, A must also be stable with respect to C^s.

More generally, when all the buyers' choice functions satisfy a condition called substitutable completability, there exists a lattice of stable outcomes that correspond to fixed points of (14.1) under a substitutable completion.[6]

Definition 14.15 [23]. A choice function C^s for seller s is *substitutably completable* if there exists a choice function $\hat{C}^s$ such that:

1. For all $Y \subseteq X$, we have that either $\hat{C}^s(Y) = C^s(Y)$ or there exists a buyer b such that $|[\hat{C}^s(Y)]_{\to b}| \geq 2$.
2. The choice function $\hat{C}^s$ is substitutable.

It turns out that substitutable completability is sufficient for the existence of a stable outcome.

Theorem 14.16 [23]. *Assume that the choice functions of all sellers are substitutably completable and buyers have unit demand. Then a stable outcome exists.*

Moreover, when the profile of choice functions is substitutably completable in such a way that each completion satisfies the law of aggregate demand, any cumulative offer mechanism is strategy-proof for buyers.

Theorem 14.17 [23]. *If every seller's choice function has a substitutable completion that also satisfies the law of aggregate demand, and buyers have unit demand, then any cumulative offer mechanism is strategy-proof for buyers.*

All unilaterally substitutable choice functions are substitutably completable [29], [46].

In fact, considerably weaker conditions are necessary and sufficient to guarantee that there is a stable mechanism that is strategy-proof for buyers; moreover, whenever a stable and strategy-proof mechanism exists, the cumulative offer mechanism is the unique stable and strategy-proof mechanism [28].[7]

[6] However, there may also exist other stable outcomes: for instance, when we use the substitutable completion (14.2), and the choice functions of the buyers are induced by

$$\succ_{\mathsf{b}(x)}\colon \{\tilde{x}\} \succ \{x\} \succ \varnothing$$
$$\succ_{\mathsf{b}(y)}\colon \{y\} \succ \varnothing,$$

the only fixed point of (14.1) corresponds to $\{\tilde{x}\}$, even though both $\{\tilde{x}\}$ and $\{x, y\}$ are stable. Note also that the full set of stable outcomes does not form a lattice in the usual way, as $\mathsf{b}(x)$ strictly prefers a different stable outcome than $\mathsf{b}(y)$.

[7] In particular, [27] showed that any when a stable and strategy-proof mechanism is guaranteed to exist, then that mechanism is equivalent to a *cumulative offer mechanism*, and in fact all cumulative offer mechanisms produce the same outcome. (The cumulative offer mechanism was defined in footnote 5.)

However, the existence of a stable outcome can be guaranteed under still weaker conditions; [20] introduced *bilateral substitutability*, which is enough to ensure that the cumulative offer mechanism produces a stable outcome. Subsequently, [28] introduced *observable substitutability across doctors*, which is necessary and sufficient to guarantee that the cumulative offer mechanism produces a stable outcome. However, observable substitutability across doctors is not the maximal domain of choice functions for which stable outcomes can be guaranteed; finding precise conditions on choice functions that ensure the existence of stable outcomes is an open problem.

14.3.2.2 Applications

The weakened substitutability conditions just discussed have been useful in real-world settings. In particular, in many-to-one matching with contracts settings, agents on the side with multiunit demand frequently have choice functions that are not substitutable and yet still allow for stable and strategy-proof matching. For instance, the US Military Academy (West Point) assigns graduating cadets to branches of service via a centralized system in which contracts encode not only the cadet and branch of service but also potential additional guaranteed years of service. Branches rank cadets according to a strict order-of-merit list but also prioritize contracts with additional guaranteed years for a fixed number of positions. As it turns out, this preference structure introduces non-substitutabilities since the offer of a contract with additional years may induce a service branch to choose a previously rejected contract; nevertheless, the choice functions of the branches are substitutably completable (and, in fact, unilaterally substitutable) and thus admit stable and strategy-proof matching [42]. This observation has led to a redesign of not only the mechanism used to assign West Point cadets but also the mechanism used to assign ROTC cadets to branches of service [5], [14]. Additionally, [35] developed a generalization of the cadet–branch matching framework, called *slot-specific priorities*, which allowed for the types of non-substitutabilities seen in the cadet-branch matching setting; this framework has proven useful in a number of real-world contexts, including school choice programs in Boston [8] and Chicago [7].

Weakened substitutability conditions were also key in the redesign of the Israeli Psychology Masters Match [18]. They have also proven fruitful in the analysis of entry-level labor markets with regional caps, such as the Japanese medical-residency matching program [30], [31], [32], the assignment of legal traineeships in Germany [6], the allocation of students to the Indian Institutes of Technology [3], and interdistrict school choice programs [17].

14.4 Supply Chains and Trading Networks

While early work on matching with contracts focused on two-sided settings, most of the key insights can be extended to a more general framework in which an agent can act as both a buyer and a seller. In [39] the two-sided setting was generalized to multi-layered supply chains, in which agents buy from agents "upstream" and sell to agents "downstream." In [12], [13], and [25] an even more general trading network setting was considered in which no *a priori* restrictions are placed on the set of possible contractual relationships.

In this section, we first consider the supply chain setting and then the more general case of trading networks. We maintain the assumptions that choice correspondences are single-valued and satisfy the irrelevance-of-rejected-contracts condition.

In supply chains and trading networks, an *intermediary* (i.e., an agent i such that there exist contracts x and y such that $\mathsf{s}(x) = i = \mathsf{b}(y)$) often sees contracts of which he is the seller and contracts of which he is the buyer as complements. The *full substitutability* condition extends (gross) substitutability to intermediaries by requiring that an intermediary consider contracts for which he is a buyer to be (gross) substitutes, contracts for which he is a seller to be (gross) substitutes, and a contract in which he is a buyer to be a (gross) complement to contracts in which he is a seller.

Definition 14.18. The choice function of agent i is *fully substitutable*, if for all $Y \subseteq X$ and all $x, z \in X$, both of the following conditions hold:

- *Same-side substitutability:* If $x, z \in X_{i\to}$ and $z \in C^i(Y \cup \{x, z\})$ then $z \in C^i(Y \cup \{z\})$. Similarly, if $x, z \in X_{\to i}$ and $z \in C^i(Y \cup \{x, z\})$ then $z \in C^i(Y \cup \{z\})$.
- *Cross-side complementarity:* If $x \in X_{i\to}$, $z \in X_{\to i}$, and $z \notin C^i(Y \cup \{x, z\})$ then $z \notin C^i(Y \cup \{z\})$. Similarly, if $x \in X_{\to i}$, $z \in X_{i\to}$, and $z \notin C^i(Y \cup \{x, z\})$ then $z \notin C^i(Y \cup \{z\})$.

Intuitively, full substitutability requires that the agents see the goods that flow through the network as gross substitutes.

14.4.1 Supply Chains

One important special case is that of networks that are (directed) acyclic. Such networks allow for "vertical" supply chain structures in which some agents intermediate-trade between agents who only buy and agents who only sell, but these networks rule out "horizontal" trade among intermediaries. Formally, we say that the economy is a *supply chain* if there do not exist contracts $x_1, \ldots, x_n$ such that $\mathsf{b}(x_\ell) = \mathsf{s}(x_{\ell+1})$ for all $1 \le \ell \le n-1$ and $\mathsf{b}(x_n) = \mathsf{s}(x_1)$.

We next extend the concept of pairwise stability to trading networks. Instead of considering blocking contracts, we consider blocks of the form of chains in the network.

Definition 14.19. An outcome $A \subseteq X$ is *chain-stable* if it is:

1. *Individually rational*: For all $i \in I$, $C^i(A) = A_i$.
2. *Chain unblocked*: There does not exist an ordered set $Z = \{z_1, \ldots, z_n\} \subseteq X \setminus A$ such that $\mathsf{s}(z_\ell) = \mathsf{b}(z_{\ell+1})$ for all $1 \le \ell \le n-1$ and such that $Z_i \subseteq C^i(Z \cup A)$ for all $i \in I$.

We now extend Theorem 14.3 to supply chains.

Theorem 14.20 [39]. *In supply chains, if the choice functions of all agents are fully substitutable then chain-stable outcomes exist.*

The proof of Theorem 14.20 is similar to the proof of Theorem 14.3. We first define aggregate choice functions C^S and C^B as follows:

$$C^S(Y, Y') \equiv \bigcup_{i \in I} C^i(Y_{i\to} \cup Y'_{\to i})_{i\to},$$
$$C^B(Y, Y') \equiv \bigcup_{i \in I} C^i(Y_{\to i} \cup Y'_{i\to})_{\to i}.$$

Here, $C^S(Y, Y')$ is the set of contracts that are chosen by their sellers when given access to the sale contracts in Y and the purchase contracts in Y'; $C^B(Y, Y')$ is defined analogously. For each $X^B, X^S \subseteq X$, we then define a generalized DA operator by

$$\Phi(X^S, X^B) = (\Phi^S(X^B, X^S), \Phi^B(X^S, X^B)),$$
$$\Phi^S(X^B, X^S) = \{x \in X : x \in C^S(X^B \cup \{x\}, X^S)\},$$
$$\Phi^B(X^S, X^B) = \{x \in X : x \in C^B(X^S \cup \{x\}, X^B)\}.$$

Proof of Theorem 14.20. The key to the proof is the following version of Lemma 14.4.

Lemma 14.21. *In supply chains, if (X^S, X^B) is a fixed point of Φ, then $A = X^S \cap X^B$ is chain-stable and $C^B(X^B, X^S) = C^S(X^S, X^B) = A$. Conversely, in supply chains in which the choice functions of all agents are fully substitutable, if A is a chain-stable outcome, then $\Phi^N(A, A)$ is a fixed point of Φ for sufficiently large N.*

Proof sketch. The proof of Lemma 14.21 is similar to the proof of Lemma 14.4, but is more subtle due to the presence of the intermediaries.

Assume first that (X^S, X^B) is a fixed point of Φ and let $A = X^S \cap X^B$. A similar argument to the proof of Lemma 14.4 shows that $C^B(X^B, X^S) = C^S(X^S, X^B) = A$, and, as in the proof of Lemma 14.4, it follows that A is individually rational. To show that A is chain-unblocked, one can inductively apply the argument from the proof of Lemma 14.4 to show that fixed points give rise to pairwise unblocked outcomes. We leave the details of this inductive argument as an exercise for the reader.

Next, let A be a chain-stable outcome and let $(X^S(n), X^B(n)) = \Phi^n(A, A)$ for positive integers n. A similar argument to the proof of Lemma 14.4 shows that $X^S(1) \cap X^B(1) = A$. One then shows inductively that $X^S(n) \cap X^B(n) = A$, that $X^S(n) \supseteq X^S(n-1)$, and that $X^B(n) \supseteq X^B(n-1)$; again we leave the details of this argument as an exercise for the reader. As X is finite, it follows that $\Phi^{|X|}(A, A)$ is a fixed point of Φ. □

Considering the ordering $\vdash$ on $X \times X$ introduced in the proof of Theorem 14.3, full substitutability implies that Φ is isotone with respect to $\vdash$. As a result, Tarski's fixed point theorem guarantees that Φ has a fixed point (X^S, X^B). By Lemma 14.21, $C^B(X^S, X^B) = C^S(X^B, X^S)$ is a stable outcome. □

We next extend the concept of stability to supply chain settings and compare it to chain stability.

Definition 14.22. An outcome $A \subseteq X$ is *stable* if it is:

1. *Individually rational*: For all $i \in I$, $C^i(A) = A_i$.
2. *Unblocked*: There does not exist a set $Z = \{z_1, \ldots, z_n\} \subseteq X \setminus A$ such that $Z_i \subseteq C^i(Z \cup A)$ for all $i \in I$.

The following straightforward extension of Theorem 14.6 shows that in supply chains in which agents have fully substitutable choice functions, if an outcome is not blocked by a chain of contracts then it is also not blocked by more general sets of contracts.

Theorem 14.23 [21]. *In supply chains, if the choice functions of all agents are fully substitutable then every chain-stable outcome is stable.*

As in the case of two-sided many-to-many markets, full substitutability comprises a maximal domain for the guaranteed existence of stable outcomes.

Theorem 14.24 [21]. *In supply chains, if the choice function of an agent i is not fully substitutable and for all distinct $j, k \in I$ there exists $(i, j, t) \in X$ for some $t \in T$, then there exist substitutable choice functions for the other agents such that no stable outcome exists.*

14.4.2 Trading Networks

To understand the role of the acyclicity assumption in Theorem 14.20, we show via example that (chain-)stable outcomes may not exist in general trading networks, even under full substitutability.

Example 14.25. There are two intermediaries i_1, i_2 and one buyer b. The set of contracts is $X = \{x, y, z\}$ and we have that $\mathsf{s}(x) = \mathsf{s}(z) = \mathsf{b}(y) = i_1$, that $\mathsf{s}(y) = \mathsf{b}(x) = i_2$, and that $\mathsf{b}(z) = b$. Note that the contracts x and y comprise a cycle. The agents' choice functions are induced by the preferences

$$\succ_{i_1}: \{y, z\} \succ_{i_1} \{x, y\} \succ_{i_1} \varnothing$$
$$\succ_{i_2}: \{x, y\} \succ_{i_2} \varnothing$$
$$\succ_b: \{z\} \succ_b \varnothing;$$

these choice functions are fully substitutable.

However, there is no chain-stable outcome. To see why, note that the only individually rational outcomes are $\{x, y\}$ and $\varnothing$. But $\{x, y\}$ blocks the outcome $\varnothing$, while $\{z\}$ blocks the outcome $\{x, y\}$.

In fact, in trading networks, it is NP-complete to determine whether a stable outcome exists as well as whether a given outcome is stable – even if all agents' choice functions are fully substitutable [11]. To analyze trading networks with cycles, we therefore consider a different extension of pairwise stability to trading networks. Under this concept, which we call trail stability, blocking contracts occur in a sequence and agents evaluate pairs of consecutive contracts in isolation rather than in reference to the entire set of blocking contracts.

Definition 14.26. An outcome $A \subseteq X$ is *trail-stable* if it is:

1. *Individually rational*: For all $i \in I$, $C^i(A) = A_i$.
2. *Trail unblocked*: There does not exist a sequence $z_1, \ldots, z_n \in X \setminus A$ of contracts such that $\mathsf{s}(z_\ell) = \mathsf{b}(z_{\ell+1})$ and $\{z_\ell, z_{\ell+1}\} \subseteq C^{\mathsf{s}(z_\ell)}(\{z_\ell, z_{\ell+1}\} \cup A)$ for all $1 \le \ell \le n-1$, $z_1 \in C^{\mathsf{s}(i_1)}(\{z_1\} \cup A)$, and $z_n \in C^{\mathsf{b}(z_n)}(\{z_n\} \cup A)$.

In supply chains, trail stability coincides with chain stability. To understand the difference between the two concepts in general trading networks, note that in Example 14.25, the outcome $\varnothing$ is a trail-stable outcome but is not chain-stable. Trail-stable outcomes turn out to exist in general trading networks under full substitutability.

Theorem 14.27 [12]. *If the choice functions of all agents are fully substitutable then trail-stable outcomes exist.*

To prove Theorem 14.27, we apply the fixed-point argument from the proof of Theorem 14.20 but use the following extension of Lemma 14.21 to trading networks.

Lemma 14.28 [1]. *If (X^S, X^B) is a fixed point of Φ. then $A = X^S \cap X^B$ is trail-stable and $C^B(X^B, X^S) = C^S(X^S, X^B) = A$. Conversely, if the choice functions of all agents are fully substitutable and A is a trail-stable outcome then $\Phi^N(A, A)$ is a fixed point of Φ for sufficiently large N.*

The proof of Lemma 14.28 is similar to the proof of Lemma 14.21, and is left as an exercise for the reader.

14.5 Transfers

Finally, we consider a setting with continuous transfers. A *contract* x is now a pair $(\omega, p_\omega) \in \Omega \times \mathbb{R}$ that specifies a bilateral trade $\omega \in \Omega$ between a *buyer* $\mathsf{b}(\omega) \in I$ and a *seller* $\mathsf{s}(\omega) \in I \setminus \{\mathsf{b}(\omega)\}$ in exchange for a monetary transfer p_ω (to be paid to the seller from the buyer). The set of possible contracts is $X \equiv \Omega \times \mathbb{R}$. A set of contracts $Y \subseteq X$ is *feasible* if it does not contain two or more contracts for the same trade: formally, Y is feasible if $(\omega, p_\omega), (\omega, \hat{p}_\omega) \in Y$ implies that $p_\omega = \hat{p}_\omega$. We call a feasible set of contracts an *outcome*. An outcome specifies a set of trades along with associated prices but does not specify prices for trades that are not in that set. We let $\tau(Y)$ be the set of trades that are associated with some contract in Y, i.e.,

$$\tau(Y) \equiv \{\psi \in \Psi : (\psi, p_\psi) \in Y \text{ for some } p_\psi \in \mathbb{R}\}.$$

An *arrangement* is a pair $[\Psi; p]$ with $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^\Omega$. Note that an arrangement specifies prices for *all* the trades in the economy.

Each agent i has a *valuation* (or *preferences*) $u_i \colon \wp(\Omega_i) \to \mathbb{R} \cup \{-\infty\}$ over the sets of trades in which they are involved, with $u_i(\varnothing) \in \mathbb{R}$; we use $u^i(\psi) = -\infty$ to denote that ψ is infeasible for i. The valuation u_i over bundles of trades gives rise to a quasilinear utility function U_i over bundles of trades and associated transfers. Specifically, for any feasible set of contracts $Y \subseteq X$, we define

$$U_i(Y) \equiv u_i(\tau(Y)) + \sum_{(\omega, p_\omega) \in Y_{i \to}} p_\omega - \sum_{(\omega, p_\omega) \in Y_{\to i}} p_\omega,$$

and, slightly abusing the notation, for any arrangement $[\Psi; p]$ we define

$$U_i([\Psi; p]) \equiv u_i(\Psi) + \sum_{\psi \in \Psi_{i\to}} p_\psi - \sum_{\psi \in \Psi_{\to i}} p_\psi.$$

The *demand correspondence* of agent i, given a price vector $p \in \mathbb{R}^\Omega$, is defined by

$$D_i(p) \equiv \arg\max_{\Psi \subseteq \Omega_i} \{U_i([\Psi; p])\}.$$

There is a natural analogue of the full substitutability condition for demand correspondences: whenever the price of an input (i.e., a trade in $\Omega_{\to i}$) increases, then i's demand for other inputs weakly increases (in the superset sense) and her supply of outputs (i.e., trades in $\Omega_{i\to}$) weakly decreases (in the subset sense); an analogous condition is required for the case where the price of an output decreases. As shown by [27], full substitutability for demand correspondences is equivalent to full substitutability for choice correspondences (as well as several other conditions). Hence, from now on we will simply say that agents' preferences are fully substitutable. We now formally define competitive equilibria for our setting.

Definition 14.29. An arrangement $[\Psi; p]$ is a *competitive equilibrium* if, for all $i \in I$,

$$\Psi_i \in D_i(p).$$

The theorem below shows that competitive equilibria exist, and are essentially equivalent to stable outcomes, when agents' preferences are fully substitutable.[8]

Theorem 14.30 [25]. *If agents' preferences are fully substitutable then competitive equilibria exist and are stable. Furthermore, for any stable outcome A, there exist prices $p_{\Omega \setminus \tau(A)}$ for the non-realized trades in $\Omega \setminus \tau(A)$ such that $[\tau(A), (p_{\tau(A)}, p_{\Omega \setminus \tau(A)})]$ is a competitive equilibrium.*

Finally, we relax the assumption that utility is perfectly transferable between agents. We suppose that instead of depending quasilinearly on net payments, utility can depend arbitrarily on the entire vector of payments. That is, we suppose that each agent i has a utility function $U_i \colon \wp(\Omega_i) \times \mathbb{R}^{\Omega_i} \to \mathbb{R} \cup \{-\infty\}$, and define utility over feasible sets Y of contracts by

$$U_i(Y) \equiv U_i(\tau(Y), t(Y)),$$

where

$$t(Y) = \begin{cases} p_\omega & \text{if } \omega \in \tau(Y)_{i\to}, \text{ where } (\omega, p_\omega) \in Y, \\ -p_\omega & \text{if } \omega \in \tau(Y)_{\to i}, \text{ where } (\omega, p_\omega) \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

This framework allows us to incorporate the possibility of frictions, such as distortionary taxes on payments on different trades. As with the quasilinear case, the full substitutability condition extends to this setting with continuous prices. In this

[8] The existence of competitive equilibria has also been shown in related exchange economy settings (see, e.g., [15], [43], [44]) in which each agent can only be a buyer or a seller.

setting, under full substitutability and regularity conditions on agents' utility functions, competitive equilibria exist and essentially coincide with trail-stable outcomes. Nevertheless, stable outcomes do not generally exist when distortionary frictions are present. Intuitively, distortions do not preclude the existence of equilibrium but they can cause equilibrium to be inefficient; however, trail-stable outcomes can often be blocked by agents who can coordinate transfers in ways that reduce the impact of the frictions.

14.5.1 Applications

One possible market design application of the trading network framework is peer-to-peer energy trading. Many energy markets around the world are shifting from large-scale centralized power generation towards inflexible, small-scale, renewable energy resources. In these energy markets, there are not only traditional suppliers (which we model as sellers) and traditional consumers (which we model as buyers), but also many "prosumers" (consumers who also generate power, e.g., using residential solar panels; we model these as intermediaries). Contracts specify a discrete quantity of energy offered by one agent to another at a given time and price. It turns out that full substitutability is a reasonable approximation of preferences of agents in such an energy market [37], [38] if, for example, economies of scale in generation are absent. Trail-stable outcomes can be computed second-by-second, thereby maintaining overall system balance without any recourse to a centralized system operator.

14.6 Exercises

Exercise 14.1 Explain why the last statement of Example 14.13 is true.

Exercise 14.2 Assume one seller has the non-substitutable preferences given in Example 14.13. Construct (multi-unit) preferences for other buyers and sellers in such a way that no stable outcome exists. (At least one buyer will have to have multi-unit demand – why?)

Exercise 14.3 Prove that if one seller has preferences given by (with $\mathsf{b}(x) \neq \mathsf{b}(y) = \mathsf{b}(y') \neq \mathsf{b}(z) \neq \mathsf{b}(x)$)

$$\{x, y, z\} \succ \{y'\} \succ \{x, z\} \succ \{x, y\} \succ \{y, z\} \succ \{x\} \succ \{z\} \succ \{y\} \succ \varnothing,$$

and if the preferences of the other sellers are substitutable, and buyers have unit-demand preferences, then a stable outcome must exist.

Exercise 14.4 Complete the proof of Lemma 14.21 or Lemma 14.28.

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