Consistency and Monotonicity in One-Sided Assignment Problems

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1 Introduction

2 Properties of solutions

3 Results

Doubles competition

Once pairs formed, partners cannot be changed during the tournament.

Passion for the sport but also pecuniary interests.

Problem: *simultaneously* decide how to form pairs and how to distribute payoffs?

Reshuffling is minor, stable solutions exist in many instances.

Doubles competition

What *properties* would the solution satisfy?

- What if some pairs drop out of the competition with their gains?
- How does a small adjustment in the estimation of potential gains affect the solution?
- What if we focus on pecuniary driven players?

Related literature

Roommate Markets:
Gale and Shapley (1962).

Two-Sided Assignment Games:
Shapley and Shubik (1972), Sasaki(1995), Toda (2005).

Roommate games with transferable utility:
Eriksson and Karlander (2001).

One-Sided Assignment Games:
Sotomayor (2005).

Partner formation problems:
Talman and Yang (2008).

One-sided assignment problems

A generic nonempty finite set of agents N.

The set of distinct *pairs* agents can form $P(N) = \{(i, j) \in N \times N \mid i \leq j\}$.

A *characteristic function* $\pi : P(N) \to \mathbb{R}_+$ such that $\pi(i, i) = 0$.

A generic *one-sided assignment problem* γ is a pair (N, π).

The *set of all one-sided assignment problems* is Γ.

One-sided assignment problems

A *matching* μ is a function $\mu : N \to N$ of order two, for each $i \in N$, $\mu(\mu(i)) = i$. $\mathcal{M}(N)$ is the *set of matchings*. At any μ, a *set of couples* $C(\mu)$ and a *set of singletons* $S(\mu)$.

μ is *optimal* if for each $\mu' \in \mathcal{M}(N)$, $\sum_{(i,j)\in\mu} \pi(i,j) \geq \sum_{(i,j)\in\mu'} \pi(i,j)$.

A *feasible outcome* is a pair (μ, u), where $u \in \mathbb{R}^{|N|}$ is a *payoff vector* such that $\sum_{i\in N} u_i = \sum_{(i,j)\in\mu} \pi(i,j)$. $\mathcal{F}(\gamma)$ is the *set of feasible outcomes*.

A feasible outcome (μ, u) is *Pareto optimal* if for each $\mu' \in \mathcal{M}(N)$, $\sum_{i\in N} u_i = \sum_{(i,j)\in\mu} \pi(i,j) \geq \sum_{(i,j)\in\mu'} \pi(i,j)$. $\mathcal{PO}(\gamma)$ is the *set of Pareto optimal outcomes*.

One-sided assignment problems

A feasible outcome (μ, u) is *individually rational* if for each $i \in N$, $u_i \geq 0$. $\mathcal{IR}(\gamma)$ is the *set of individually rational outcomes*.

A feasible outcome (μ, u) is *couple rational* if for each $(i, j) \in C(\mu)$, $u_i + u_j \geq \pi(i, j)$. $\mathcal{CR}(\gamma)$ is the *set of couple rational outcomes*.

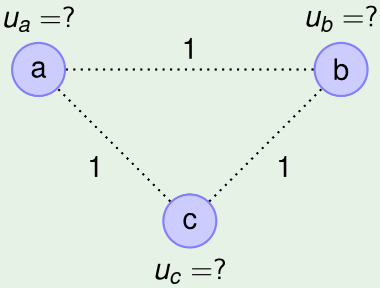
Let $\gamma \in \Gamma$ and $(\mu, u) \in \mathcal{F}(\gamma)$. If there are two agents $(i, j) \in P(N)$ such that $i \neq j$ and $u_i + u_j < \pi(i, j)$, then $\{i, j\}$ is a *blocking pair* for the outcome (μ, u).

A feasible outcome (μ, u) is *stable* if it is individually rational and no blocking pairs exist. $\mathcal{S}(\gamma)$ is the *set of stable outcomes*.

Let $\gamma \in \Gamma$ be *solvable* if $\mathcal{S}(\gamma) \neq \emptyset$.

The set of stable outcomes may be empty

Example 1. A problem that is not solvable.



For each $(\mu, u) \in \mathcal{F}(\gamma)$, $\exists$ i,j s.t. $\{i, j\} \subseteq N$, $i \neq j$, and $u_i + u_j < \pi(i, j)$.

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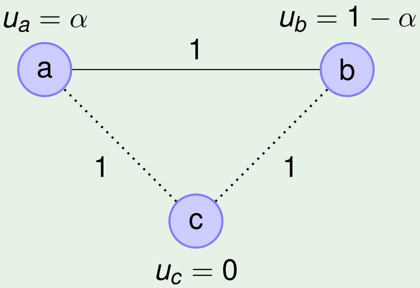

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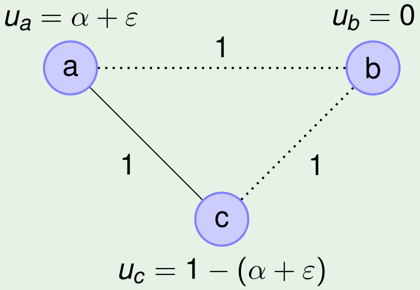

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One-sided assignment problems

Remark 1.

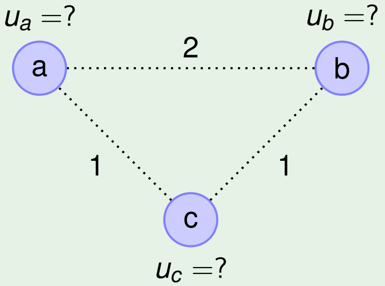
On the class of solvable games, the set of stable outcomes equals the core, i.e., for all $\gamma \in \Gamma$, $\mathcal{S}(\gamma) = \mathcal{C}(\gamma)$.

A *solution* φ is a correspondence that associates with each $\gamma \in \Gamma$ a non-empty subset of feasible outcomes, i.e., for each $\gamma \in \Gamma$, $\varphi(\gamma) \subseteq \mathcal{F}(\gamma)$ and $\varphi(\gamma) \neq \emptyset$.

A solution φ' is a *subsolution* of solution φ if for each $\gamma \in \Gamma$, $\varphi'(\gamma) \subseteq \varphi(\gamma)$.

Can we always map a solvable one-sided problem to a two-sided problem?

Example 2. The problems are *not* core-isomorphic.



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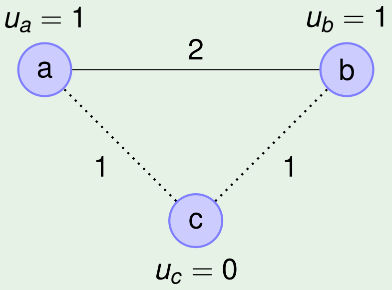

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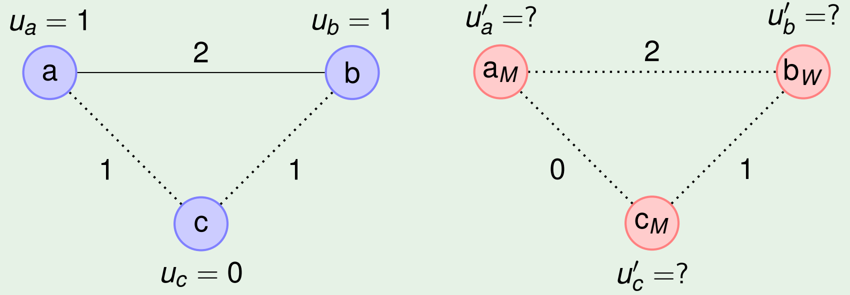

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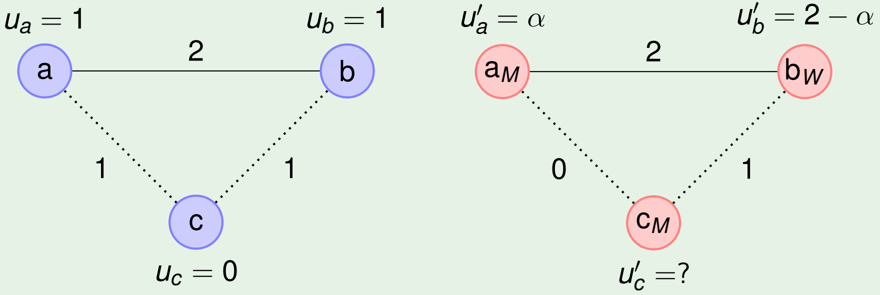


Where $\alpha \in [0, 1]$. Thus, $\mathcal{S}(\gamma) \subsetneq \mathcal{S}(\gamma')$.

Properties of solutions

Individual rationality: for each $\gamma \in \Gamma$, $\varphi(\gamma) \subseteq \mathcal{IR}(\gamma)$.

Couple rationality: for each $\gamma \in \Gamma$, $\varphi(\gamma) \subseteq \mathcal{CR}(\gamma)$.

Pareto optimality: for each $\gamma \in \Gamma$, $\varphi(\gamma) \subseteq \mathcal{PO}(\gamma)$.

Pareto indifference: for each $\gamma \in \Gamma$ and each $(\mu, u) \in \varphi(\gamma)$, if $(\mu', u) \in \mathcal{F}(\gamma)$, then $(\mu', u) \in \varphi(\gamma)$.

For $\gamma \in \Gamma$, $\mu \in \mathcal{M}(N)$, i and j are *dummy agents* for μ if $\pi(i, j) = 0$ and $(i, j) \in C(\mu)$. $DA(\gamma, \mu)$ is the *set of dummy agents* for μ.

Indifference with respect to unmatching dummy pairs: for each $\gamma \in \Gamma$, each $(\mu, u) \in \varphi(\gamma)$, and all $(\mu', u) \in \mathcal{F}(\gamma)$ such that $C(\mu') \subseteq C(\mu)$ and $S(\mu') \setminus S(\mu) \subseteq DA(\gamma, \mu)$, $(\mu', u) \in \varphi(\gamma)$.

Properties of solutions

Continuity: for each $N \in \mathcal{N}$, each $\mu \in \mathcal{M}(N)$, each sequence $\{\pi^k\}_{k\in\mathbb{N}}$, and each sequence $\{u^k\}_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $\pi^k \in \Pi(N)$ and $(\mu, u^k) \in \varphi(N, \pi^k)$, if $(N, \pi^k) \xrightarrow[k\to\infty]{} (N, \pi)$ and $u^k \xrightarrow[k\to\infty]{} u$, then $(\mu, u) \in \varphi(N, \pi)$.

Consistency: for each $\gamma \in \Gamma$, each $(\mu, u) \in \varphi(\gamma)$, and each $N' \subseteq N$ such that $\mu(N') = N'$, $(\mu_{|N'}, u_{|N'}) \in \varphi(\gamma_{|N'})$.

Properties of solutions

Weak pairwise-monotonicity: for each $N \in \mathcal{N}$, each $\pi \in \Pi(N)$, each $(i,j) \in P(N)$, $i \neq j$, and each $\pi^* \in \Pi(N)$ such that

$$\pi^*(i,j) \geq \pi(i,j) \quad \text{and} \tag{1}$$
$$\pi^*(i',j') = \pi(i',j'), \quad \text{otherwise,} \tag{2}$$

if $(\mu, u) \in \varphi(N, \pi)$, then there exists $(\mu^*, u^*) \in \varphi(N, \pi^*)$ such that $u_i^* + u_j^* \geq u_i + u_j$.

Related literature

Sasaki (1995), gives two characterizations of the core:

1. *consistency*, individual rationality, couple rationality, Pareto optimality, weak pairwise-monotonicity and continuity (Theorem 2)
2. replace continuity by Pareto indifference (Theorem 4).

Sasaki (1995), proves both characterizations by showing that:

1. the core satisfies all properties used.
2. a solution that satisfies all properties is a subsolution of the core.
3. a solution that is a subsolution of the core and satisfies all properties equals the core.

Positive results on the class of solvable problems

Proposition 1.

On the class of solvable one-sided assignment problems, the core satisfies individual rationality, couple rationality, Pareto optimality, Pareto indifference, indifference with respect to unmatching dummy pairs, continuity, consistency, and weak pairwise-monotonicity.

Remark 2.

The class of solvable problems is topologically *closed*.

Positive results on the class of solvable problems

Theorem 1.

On the class of solvable problems, if φ is a subsolution of the core satisfying indifference with respect to unmatching dummy pairs, continuity, and consistency, then φ coincides with the core.

Show Proof Sketch of Theorem 1

Remark 3. indifference with respect to unmatching dummy pairs and Sasaki (1995) Theorem 1.

For Theorem 1 to hold, the requirement that the subsolution of the core φ satisfies indifference with respect to dummy agents is necessary.

Positive results on the class of solvable problems

What if we extend π s.t. $\pi : P(N) \to \mathbb{R}_+$ and we do *not* require that for each $i \in N$, $\pi(i, i)$ is fixed to 0?

Theorem 2.

On the class of solvable one-sided problems with nonnegative reservation values that are allowed to vary, if φ is a subsolution of the core satisfying continuity and consistency, then φ coincides with the core.

Theorem 3.

On the class of solvable problems, if φ is a subsolution of the core satisfying consistency and Pareto indifference, then φ coincides with the core.

Impossibilities and limitations

Theorem 4.

There exists no solution φ that coincides with the core whenever the core is nonempty and that satisfies consistency.

▸ Show Proof Sketch of Theorem 4

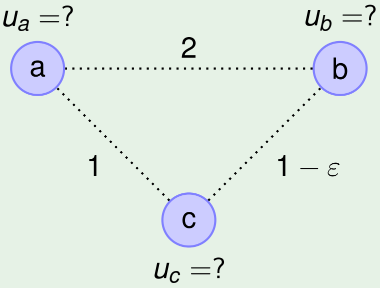
Corollary 1.

(a) There exists no solution φ that is subsolution of the core whenever the core is nonempty and that satisfies indifference with respect to unmatching dummy pairs, continuity, and consistency.

(b) There exists no solution φ that is subsolution of the core whenever the core is nonempty and that satisfies Pareto indifference and consistency.

Impossibilities and limitations

Example 3. Changes of the core when the value of a couple changes.



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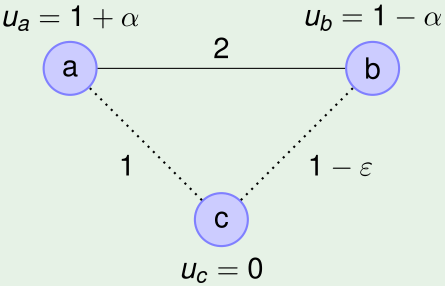

$\mathcal{S}(\gamma) = \{(\mu, u) \mid \mu = (b, a, c)$ and $u = (1 + \alpha, 1 - \alpha, 0)$ for $\alpha \in [0, \varepsilon]\}$.

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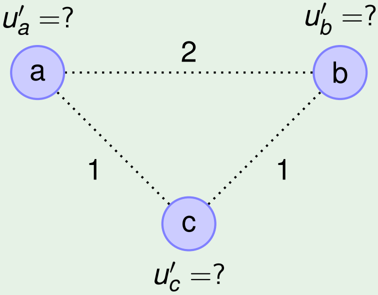

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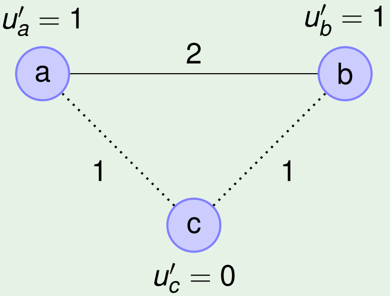

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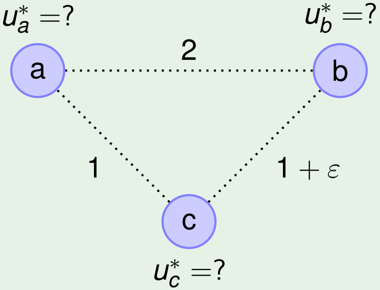

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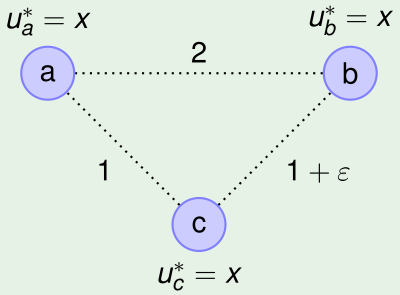

$\mathcal{S}(\gamma) = \{(\mu, u) \mid \mu = (b, a, c)$ and $u = (1 + \alpha, 1 - \alpha, 0)$ for $\alpha \in [0, \varepsilon]\}$.
$\mathcal{S}(\gamma') = \{(\mu', u') \mid \mu' = (b, a, c)$ and $u' = (1, 1, 0)\}$.
$\mathcal{S}(\gamma^*) = \emptyset$.

Thanks for your time and attention!

The slides of this talk will be available at:

`http://www.nichifor.net`

Consistency and Monotonicity in One-Sided Assignment Problems

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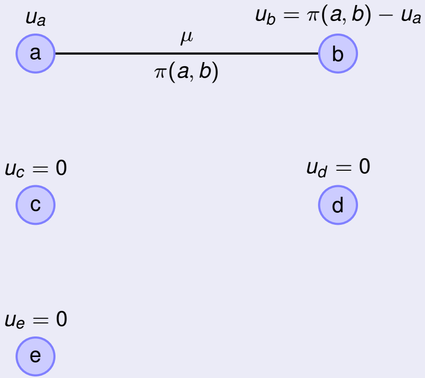
Proof of Theorem 1

Th 1: if $\varphi(\gamma) \subseteq \mathcal{S}(\gamma)$, i.u.d.p., cont. and cons. $\Rightarrow \varphi(\gamma) \supseteq \mathcal{S}(\gamma)$.



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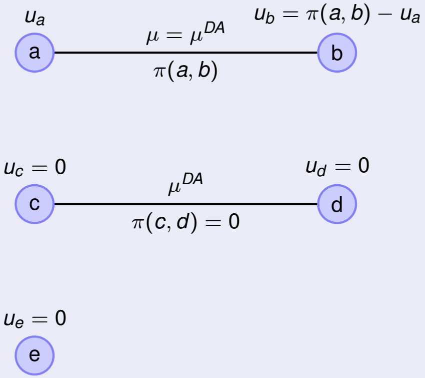

Let $(\mu, u) \in \mathcal{S}(\gamma)$. From μ and a maximal number of dummy agents $\Rightarrow \mu^{DA}$.

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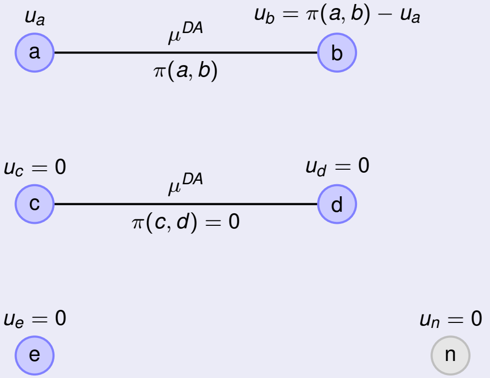

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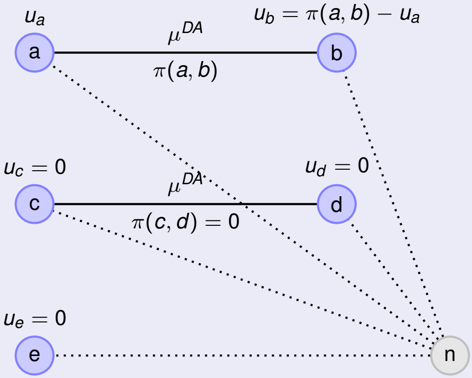

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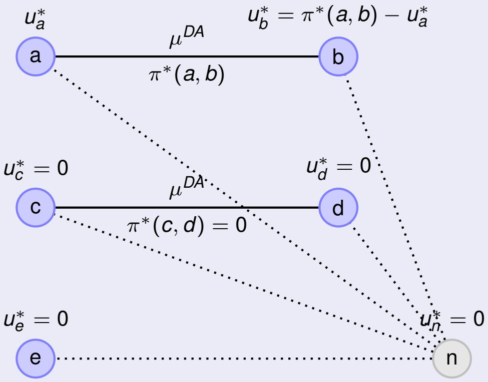

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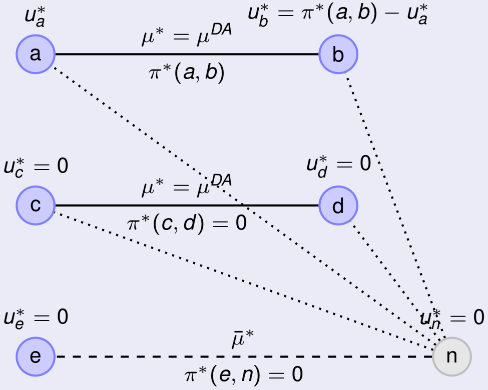

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$\forall i \in N \setminus \{e\}, \mu^*(i) = \mu^{DA}(i)$; $\mu^*(e) = e$, $\mu^*(n) = n$

$\forall i \in N \setminus \{e\}, \bar{\mu}^*(i) = \mu^{DA}(i)$; $\bar{\mu}^*(e) = n$.

$(\mu^*, u^*), (\bar{\mu}^*, u^*) \in \mathcal{S}(\gamma^*)$.

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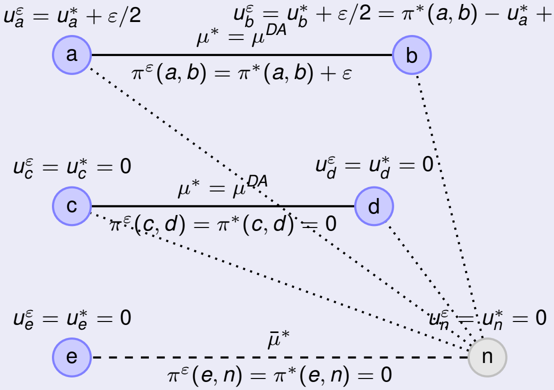

Construct:

$\forall (i,j) \in C(\mu^{DA})$,
$[\pi^\varepsilon(i,j) = \pi^*(i,j) + \varepsilon$,
$u_i^\varepsilon = u_i^* + \frac{\varepsilon}{2}$, $u_j^\varepsilon = u_j^* + \frac{\varepsilon}{2}]$;
$\pi^\varepsilon(d,n) = \pi^*(d,n)$,
$u_d^\varepsilon = u_n^\varepsilon = 0$.

$\forall (i,j) \in P(N) \setminus C(\mu^{DA})$,
$\pi^\varepsilon(i,j) = \pi^*(i,j)$.

Let $\gamma^\varepsilon = (N^*, \pi^\varepsilon)$.

$(\mu^*, u^\varepsilon), (\bar{\mu}^*, u^\varepsilon) \in \mathcal{S}(\gamma^\varepsilon) \Rightarrow$
$\{\mu^*, \bar{\mu}^*\} \subseteq \mathcal{OM}(\gamma^\varepsilon)$.

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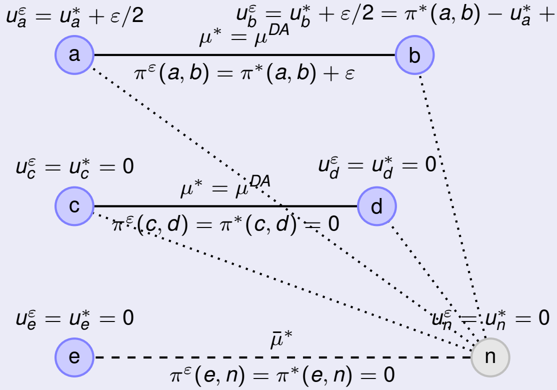

Claim 1: $\{\mu^*, \bar{\mu}^*\} = \mathcal{OM}(\gamma^\varepsilon)$.

Claim 2: Let $(\mu^*, \tilde{u}) \in \mathcal{S}(\gamma^\varepsilon)$.

Then, $\forall i \in N$, $|\tilde{u}_i - u_i^\varepsilon| \leq \frac{\varepsilon}{2}$.

By assumption, $\varphi(\gamma^\varepsilon) \subseteq \mathcal{S}(\gamma^\varepsilon)$.
Thus, $\mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon) \neq \emptyset$.

Suppose
$(\bar{\mu}^*, \bar{u}) \in \mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon)$,
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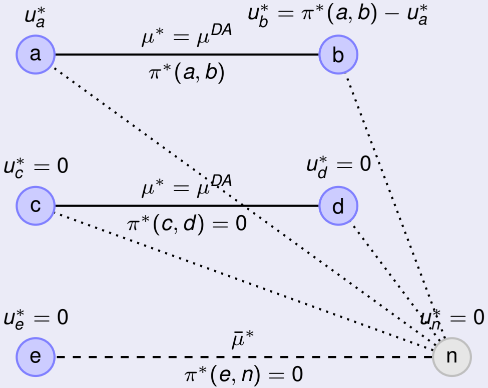

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By Claim 2 & letting $\varepsilon \to 0$,
$\forall i \in N$, $|\bar{u}_i - u_i^\varepsilon| \to 0$, $u_i^\varepsilon \to u_i^*$.
$\Rightarrow^{cont.}$: $(\mu^*, u^*) \in \varphi(\gamma^*)$.

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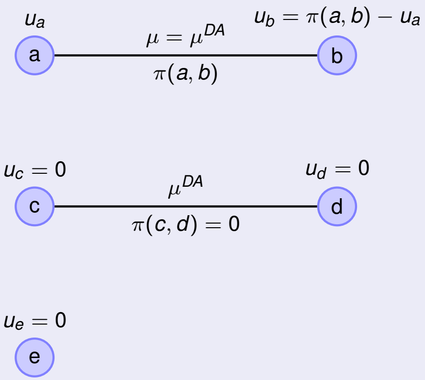

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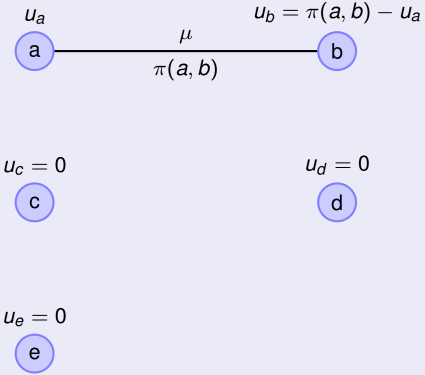
$\Rightarrow^{cont.}$: $(\mu^*, u^*) \in \varphi(\gamma^*)$.

$\Rightarrow^{cons.}$: $(\mu^*_{|N}, u^*_{|N}) \in \varphi(\gamma^*_{|N})$,
i.e., $(\mu^{DA}, u) \in \varphi(\gamma)$.

Proof of Theorem 1

Th 1: if $\varphi(\gamma) \subseteq \mathcal{S}(\gamma)$, i.u.d.p., cont. and cons. $\Rightarrow \varphi(\gamma) \supseteq \mathcal{S}(\gamma)$.



Claim 1: $\{\mu^*, \bar{\mu}^*\} = \mathcal{OM}(\gamma^\varepsilon)$.

Claim 2: Let $(\mu^*, \tilde{u}) \in \mathcal{S}(\gamma^\varepsilon)$.

Then, $\forall i \in N$, $|\tilde{u}_i - u_i^\varepsilon| \leq \frac{\varepsilon}{2}$.

By assumption, $\varphi(\gamma^\varepsilon) \subseteq \mathcal{S}(\gamma^\varepsilon)$.
Thus, $\mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon) \neq \emptyset$.

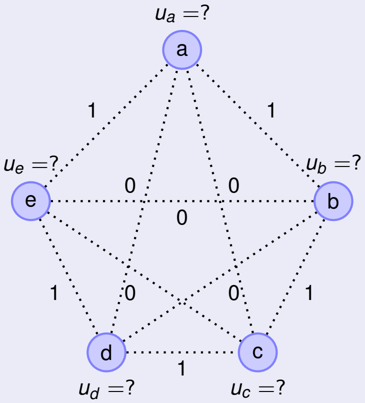
Suppose
$(\bar{\mu}^*, \bar{u}) \in \mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon)$,
$\Rightarrow^{i.u.d.p.}$,
$(\mu^*, \bar{u}) \in \mathcal{S}(\gamma^\varepsilon) \cap \varphi(\gamma^\varepsilon)$.

By Claim 2 & letting $\varepsilon \to 0$,
$\forall i \in N$, $|\bar{u}_i - u_i^\varepsilon| \to 0$, $u_i^\varepsilon \to u_i^*$.
$\Rightarrow^{cont.}$: $(\mu^*, u^*) \in \varphi(\gamma^*)$.

$\Rightarrow^{cons.}$: $(\mu^*_{|N}, u^*_{|N}) \in \varphi(\gamma^*_{|N})$,
i.e., $(\mu^{DA}, u) \in \varphi(\gamma)$.
Since $(\mu, u) \in \mathcal{F}(\gamma)$,
$\Rightarrow^{i.u.d.p.}$: $(\mu, u) \in \varphi(\gamma)$. ◂ Back

Proof of Theorem 4

Th 4: $\nexists\ \varphi$ s. t. $\varphi(\gamma) = \mathcal{S}(\gamma)$ whenever $\mathcal{S}(\gamma) \neq \emptyset$ & φ is cons.



Proof of Theorem 4

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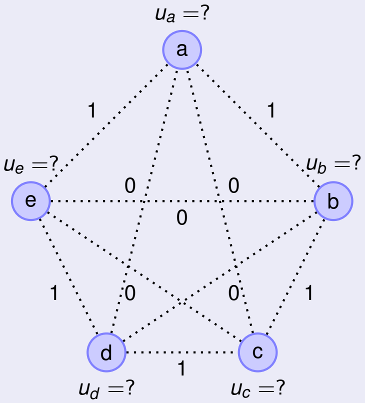

$\mathcal{S}(\gamma) = \{\emptyset\}$.

Proof of Theorem 4

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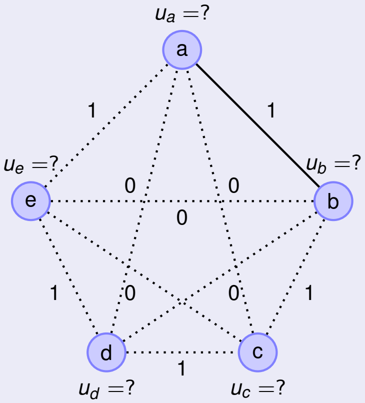

Case 1: Let $(\mu, u) \in \varphi(\gamma)$ s.t.
$[|C(\mu)| = 0$ & $|S(\mu)| = 5]$ or
$[|C(\mu)| = 1$ & $|S(\mu)| = 3]$.

Proof of Theorem 4

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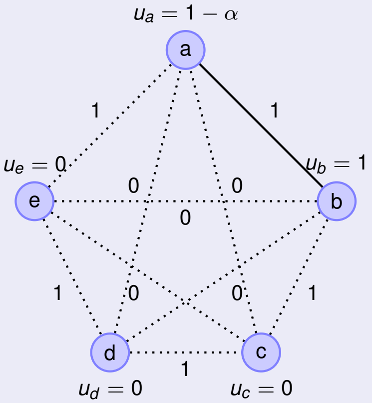

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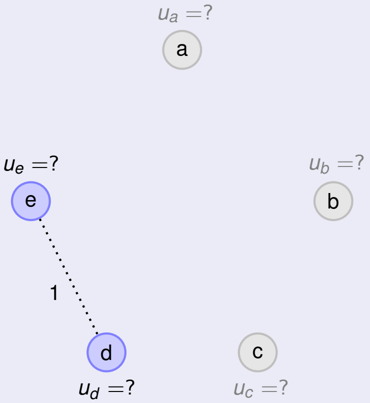

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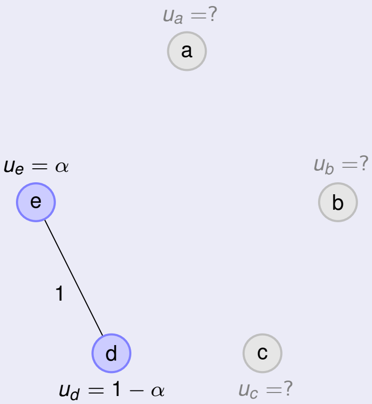

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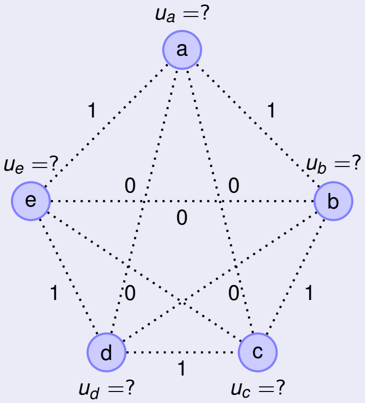
$[|C(\mu)| = 1$ & $|S(\mu)| = 3]$. $\rightarrow u_d = u_e = 0$.

Since $\mathcal{S}(\gamma_{|N'}) \neq \emptyset$, $\varphi(\gamma_{|N'}) = \mathcal{S}(\gamma_{|N'})$.

Hence, in contradiction to φ being consistent, $(\mu_{|N'}, u_{|N'}) \notin \varphi(\gamma_{|N'})$.

Proof of Theorem 4

Th 4: $\nexists\ \varphi$ s. t. $\varphi(\gamma) = \mathcal{S}(\gamma)$ whenever $\mathcal{S}(\gamma) \neq \emptyset$ & φ is cons.



Case 2:
$(\mu, u) \in \varphi(\gamma)$ s.t. $|C(\mu)| = 2$ & $|S(\mu)| = 1$.

Proof of Theorem 4

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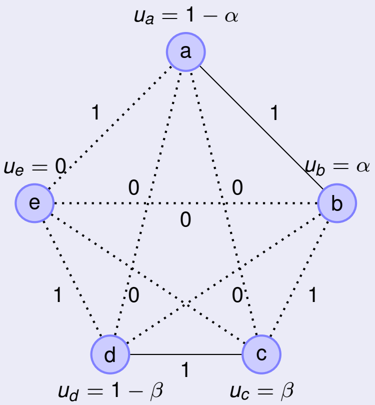

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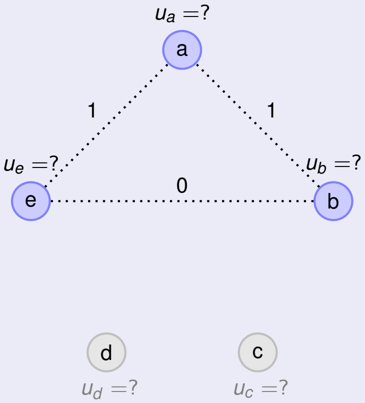

Case 2:
$(\mu, u) \in \varphi(\gamma)$ s.t. $|C(\mu)| = 2$ & $|S(\mu)| = 1$.

Step 1: $N' = \{a, b, e\}$.

Proof of Theorem 4

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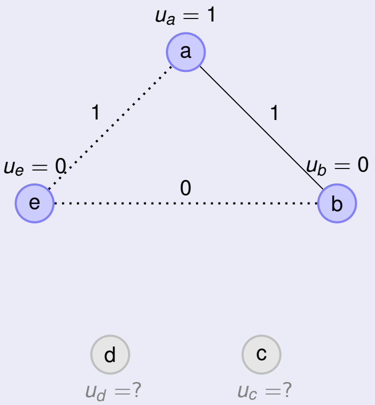

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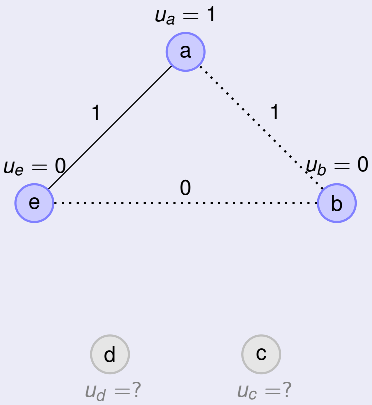

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Step 1: $N' = \{a, b, e\}$. $\rightarrow u_a = 1$.

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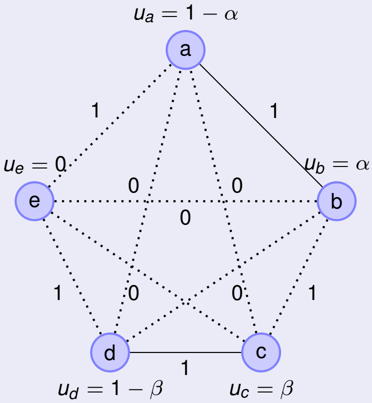

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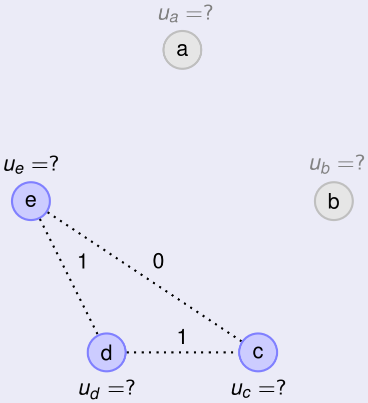

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Step 2: $N'' = \{c, d, e\}$.

Proof of Theorem 4

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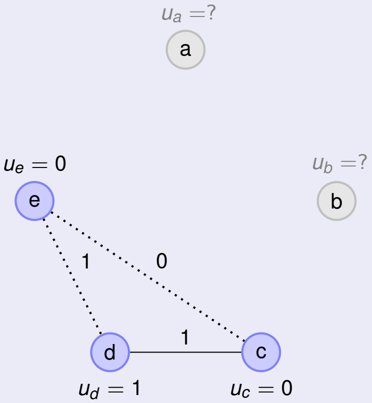

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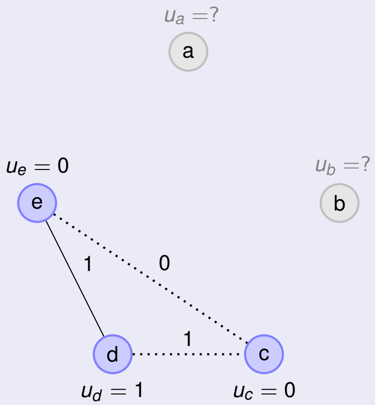

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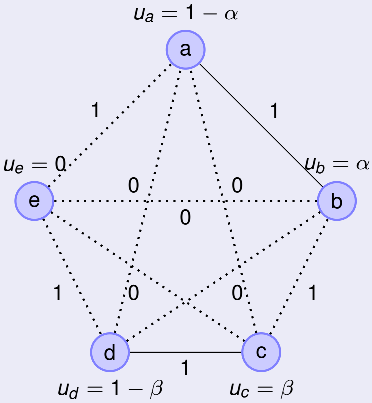

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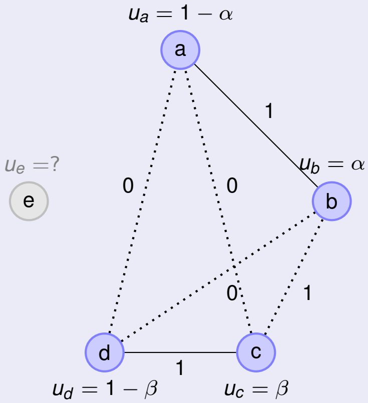

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Step 3: $N^* = \{a, b, c, d\}$.

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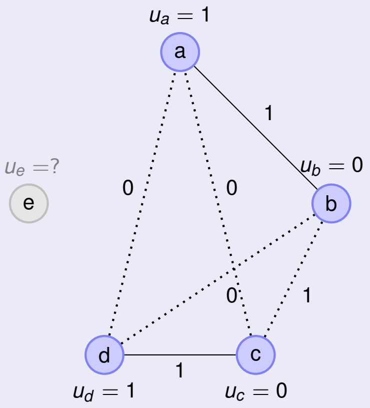

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Step 3: $N^* = \{a, b, c, d\}$. By consistency: $u_a = u_d = 1$. $\rightarrow u_b = u_c = 0$.

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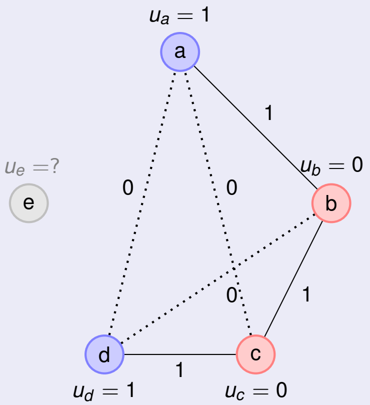

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Step 3: $N^* = \{a, b, c, d\}$. By consistency:
$u_a = u_d = 1$. $\rightarrow u_b = u_c = 0$.

$\rightarrow (b, c)$ blocking pair for $(\mu|_{N^*}, u|_{N^*})$,
contradicting $\varphi(\gamma|_{N^*}) = \mathcal{S}(\gamma|_{N^*})$. Back